Refined Theories for Beam Bending: A Simplified Approach to Structural Analysis

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ABSTRACT

This study develops a refined beam theory that improves upon classical models by accurately capturing transverse shear deformation without requiring shear correction factors. The proposed approach maintains the simplicity of the Bernoulli-Euler theory while achieving higher precision in predicting transverse deflections, axial stresses, and shear stresses. A linearly elastic, homogeneous, and isotropic material with a uniform rectangular cross-section is assumed. The accuracy of the proposed theory is validated through comparisons with advanced shear deformation theories, showing that it provides reliable results with reduced computational complexity. Furthermore, the theory's applicability is demonstrated through case studies, showcasing its effectiveness in practical structural design and analysis Numerical comparisons indicate minimal percentage differences, with a maximum deviation of -0.37% for simply supported beams and -0.82% for fully clamped beams in transverse deflection predictions. The results align well with advanced shear deformation theories and two-dimensional elasticity solutions, confirming the model's reliability. This theory enhances structural analysis, particularly for thick and shear-deformable beams, with potential extensions to anisotropic materials, dynamic loading, and complex boundary conditions in future research.

Keywords-refined beam theory; structural analysis; transverse displacement; axial stress; transverse shear stress; thick beams; analytical solutions

I. INTRODUCTION

The analysis of beam bending is a key characteristic of structural mechanics and engineering design, forming the basis for understanding the behavior of beams under various loading conditions. Classical theories, such as these developed by Euler-Bernoulli and Timoshenko, have long provided robust frameworks for analyzing bending mechanics [1-2]. However, these theories are built on simplifying assumptions, including the neglect of transverse shear deformation and rotational inertia effects. While effective for slender beams, these limitations reduce their applicability to thick beams or materials with complex mechanical properties, highlighting the need for more advanced models. To address these challenges, researchers have developed the Higher-Order Shear Deformation Theories (HSDTs). Compared to classical theories, HSDTs incorporate additional terms to capture the effects of transverse shear deformation more accurately. Significant advancements include the parabolic shear deformation theories proposed by [3-8]. These theories provide a better way to understand how beams bend without needing extra adjustments for shear stress at the top and bottom surfaces. This leads to a more accurate way of modeling how beams deform.

Transverse shear deformation is important when analyzing thick or slender beams. Unlike thin beams, shear deformation can significantly increase deflections and reduce buckling loads and vibrational frequencies. To measure these effects, it is essential to employ nondimensionalization, as mentioned by authors in [9]. Furthermore, studies such as [10-12] emphasize the critical role of transverse shear in accurately predicting deformation behavior. To address the limitations of first-order shear deformation theories, researchers have developed numerous HSDTs [13-15], providing more accurate estimations of beam deformation while they introduce additional equations and unknown functions, increasing computational demands. Authors in [16], noted that the complexity of these models can be an obstacle to their practical application, emphasizing the importance of balancing accuracy and computational efficiency. Based on this, in [17] a hyperbolic shear deformation theory for static and dynamic analysis was proposed, while in [18-20] trigonometric models were applied to analyze flexural behavior under various loading and support conditions. A noteworthy contribution to the field is the research of authors in [21], who developed an analytical model for static bending of thick isotropic rectangular beams with diverse boundary conditions. They utilized a combination of Fourier series and shear deformation theories, to find solutions for stress and displacement. However, their study did not cover some areas, like the bending and shear stress at the built-in ends, indicating the potential of future work.

Comprehensive reviews by [22-23] highlight the domination of Navier-type, closed-form solution for simply supported beams in the literature. Analytical solutions for beams with built-in ends or specialized loading conditions remain limited, emphasizing the need for continued innovation in this domain. Alternative methodologies have also developed, such as splitting transverse displacement into sub-components. Authors in [24] introduced a system of two governing equations, inertially coupled in dynamic cases and decoupled for static problems. Similar approaches performed in [25-28], offering a fresh perspective on addressing shear deformation. Significant research has been conducted to develop finite elements resistant to shear locking, as supported in [29-30]. Recent studies highlight the continuing evolution of shear deformation theories, focusing their importance in modern structural analysis [31-37]. These advancements ensure that the field remains active, with researchers continuously improving models and broaden their use to solve complex engineering problems.

Future research can expand this theory for wider use. It provides a solid foundation for analyzing the behavior of beams on an elastic foundation [38-40]. Its application can help address complex structural challenges, such as accounting for foundation stiffness and interaction effects, while maintaining computational efficiency. Additionally, it can be adapted to study more advanced cases, including non-uniform foundations or dynamic loading conditions

The purpose of this work is to develop an advanced theory that addresses the gaps of classical theory and presents fundamental relationships in a simple and practical form for implementation. The derivations are focused on plane elements with rectangular cross-sections. However, by adjusting the geometric characteristics, the theory can be applied to elements with arbitrary cross-sections. Analytical solutions for the static bending of beams with built-in boundary conditions are derived. To demonstrate the effectiveness of the proposed theory, illustrative examples of the static bending of shear-deformable isotropic rectangular beams are provided. The numerical results are compared with other refined theories to validate the accuracy and reliability of the proposed approach.

II. ASSUMPTIONS UNDERLYING THE THEORY

The assumptions underlying the theoretical formulation of the proposed model are as follows:

- Assumption 1:

The beam under investigation, illustrated in Figure 1, is located in the $0 - x_1 - x_2 - x_3$ Cartesian coordinate system and spans the region:

$$0 \le x_1 \le l,\ -\frac{b}{2} \le x_2 \le \frac{b}{2},\ -z \le x_3 \le z,\ z = \frac{h}{2}$$

where x_1, x_2, and x_3 represent the Cartesian coordinates, while l and b denote the beam's length and width in the x_1 and x_2 directions, respectively. The thickness of the beam along the x_3-axis is represented by h.

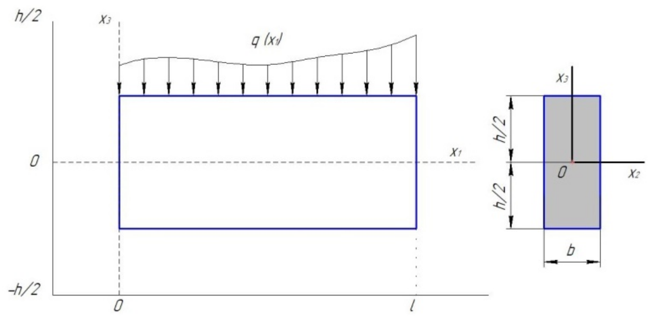

Fig. 1. Geometry of the beam and the coordinate system.

- Assumption 2:

The beam consists of a homogeneous, linearly elastic, isotropic material and the applied loads are static and uniformly or non-uniformly distributed along the beam.

- Assumption 3:

The beam's boundary conditions are defined at the ends x_1 = 0 and x_1 = l, where variationally consistent conditions are applied.

- Assumption 4:

The deformation and stress distribution along the x_3-axis follow predefined laws, ensuring consistency with the refined beam theory framework.

- Assumption 5:

The transverse shear deformation varies according to a prescribed function, capturing its dependence on the thickness coordinate.

- Assumption 6:

The horizontal displacement component U_1 does not experience any tensile or compressive deformation, maintaining the integrity of the beam's axial behavior.

III. CURRENT THEORY: DISPLACEMENT FUNCTION, STRAINS, STRESSES, CROSS-SECTIONAL BENDING MOMENT, AND SHEARING FORCE

A. Theory Development

The stress-strain state in the exact formulation is defined by the following relationships of elasticity theory [1, 2]. Equilibrium equations expressed in terms of stress components:

$$\begin{aligned} &\frac{\partial \sigma_1}{\partial x_1}+\frac{\partial \tau_{13}}{\partial x_3}+X=0 \\ &\frac{\partial \tau_{13}}{\partial x_1}+\frac{\partial \sigma_3}{\partial x_3}+Z=0 \end{aligned} \qquad (1)$$

where: σ_1, σ_3 are the normal stresses directed along the coordinate axes x_1, x_3, τ_{13} is shear stress perpendicular to these axes *X, Z* are the components of the body force acting along the coordinate planes.

Kinematic relations (Cauchy):

$$\varepsilon_1=\frac{\partial U_1}{\partial x_1},\ \varepsilon_3=\frac{\partial U_3}{\partial x_3},\ \gamma_{13}=\frac{\partial U_1}{\partial x_3}+\frac{\partial U_3}{\partial x_1} \qquad (2)$$

where $\varepsilon_1, \varepsilon_3$ are the linear strains, γ_{13} is the shear strain, and U_1, U_3 are the displacement components along the coordinate axes x_1, x_3

Physical relations (Hooke's law):

$$\begin{aligned} &\varepsilon_1=\frac{1}{E}(\sigma_1-\nu\sigma_3),\quad \varepsilon_3=\frac{1}{E}(\sigma_3-\nu\sigma_1) \\ &\gamma_{13}=\frac{\tau_{13}}{G},\ G=\frac{E}{2(1+\nu)} \end{aligned} \qquad (3)$$

where *E* is the modulus of elasticity of the material, and ν is Poisson's ratio. This theory is based on the integral characteristics of stresses and displacements:

$$\begin{aligned} &M=\int_{-z}^{z}\sigma_1 x_3 dx_3,\quad Q=\int_{-z}^{z}\tau_{13}dx_3 \\ &M_\sigma=\int_{-z}^{z}\sigma_3 x_3 dx_3,\ N=\int_{-z}^{z}\sigma_1 dx_3 \\ &N_\sigma=\int_{-z}^{z}\sigma_3 dx_3,\quad W_0=\frac{1}{h}\int_{-z}^{z}U_3 dx_3 \\ &\theta=\frac{U_1^+-U_1^-}{h}=\frac{12}{h^3}\int_{-z}^{z}(U_1\cdot x_3)dx_3 \end{aligned} \qquad (4)$$

where M_σ, N_σ are the bending moment and axial force due to normal stress σ_3, *N* is the axial force, W_0 and θ are the normal displacement and the angle of cross-sectional rotation, U_1^+, U_1^- are the horizontal displacements $U_1(x_1, x_3)$ along the $x_3 = z$ and $x_3 = -z$ axes, respectively.

By integrating (1) and considering (4) in the absence of body forces, the equilibrium equations are obtained in terms of internal forces:

$$\frac{dM}{dx_1}-Q=0,\quad \frac{dQ}{dx_1}+q=0,\quad q=\sigma_3\Big|_{-z}^{z} \qquad (5)$$

Thus, from Hooke's law (3) and the notations (4), the following expressions are derived:

$$\begin{aligned} &M=EJ\frac{d\theta}{dx_1}+\nu M_\sigma,\ Q=A(\theta+\frac{dW_0}{dx_1}) \\ &J=\frac{bh^3}{12},\quad A=Gh \end{aligned} \qquad (6)$$

where *EJ* represents the flexural rigidity, while *A* denotes the shear rigidity.

Substituting (6) into the first equation of system (5) we get:

$$EJ\frac{d^2\theta}{dx_1^2}+\nu\frac{dM_\sigma}{dx_1}-A(\theta+\frac{dW_0}{dx_1})=0 \qquad (7)$$

Equation (7) can be rewritten in operator form:

$$\begin{aligned} &L(\theta)-\frac{d}{dx_1}(P)=0 \\ &L(\theta)=\frac{EJ}{A}\frac{d^2\theta}{dx_1^2}-\theta,\quad P=W_0-\nu\frac{M_\sigma}{A} \end{aligned} \qquad (8)$$

where *L* is a linear operator.

The solution of (8) can be expressed as follows:

$$\theta=-\frac{\frac{dP}{dx_1}}{L}$$

From here, the components of the displacements are obtained:

$$\begin{aligned} &P=W-\frac{EJ}{A}\frac{d^2W}{dx_1^2} \\ &\theta=-\frac{dW}{dx_1} \end{aligned} \qquad (9)$$

From the second equation of (5), considering (6) and (9), the governing equation for determining the function $W(x_1)$ is derived:

$$EJ\frac{d^4W}{dx_1^4}=q+\nu\frac{d^2M_\sigma}{dx_1^2} \qquad (10)$$

Integrating the unused Hooke's law (3), considering (4), it can be obtained:

$$U_3^+ + U_3^- = 2W_0-\frac{2J\nu}{h}\frac{d\theta}{dx_1}+\frac{2\left(1-\nu^2\right)}{Eh}M_\sigma \qquad (11)$$

Another integral of the same law can be written as:

$$\int_{-z}^{z}\varepsilon_3 dx_3=\frac{1}{E}\int_{-z}^{z}(\sigma_3-\nu\sigma_1)dx_3$$
$$U_3^+-U_3^-=\frac{1}{E}\left(N_\sigma-\nu N\right) \quad (12)$$

where U_3^+, U_3^- are the vertical displacements $U_3(x_1,x_3)$ along the $x_3 = z$ and $x_3 = -z$ axes, respectively.

From (11) and (12), the displacements U_3^+ and U_3^- are determined. Within the framework of the classical Euler-Bernoulli beam theory, the stresses can be represented as follows [1]:

$$\sigma_1=\frac{x_3}{J}M,\quad \sigma_3=\delta(x_3)q,\quad \tau_{13}=\frac{h^2 f(x_3)}{J}Q$$
$$f(x_3)=\frac{1}{8}-\frac{1}{2}\frac{x_3^{\,2}}{h^2},\quad \delta(x_3)=\frac{1}{2}+\frac{3}{2}\cdot\frac{x_3}{h}-2\frac{x_3^{\,3}}{h^3} \quad (13)$$

where $f(x_3)$ is the transverse shear stress distribution function, $\delta(x_3)$ is the normal transverse stress distribution function. Assuming the stress σ_3 in the form of (13):

$$M_\sigma=\frac{h^2}{10}\cdot q$$
$$N_\sigma=\frac{qh}{2}$$

B. The Displacement Field

Based on the aforementioned assumptions and results, the displacement field of the present beam theory is defined as follows:

$$U_1(x_1,x_3)=-x_3\frac{dW}{dx_1}=x_3\theta,$$
$$U_3^+(x_1)=W(x_1)+\frac{J}{h}(\nu-\frac{Eh}{A})\frac{d^2W}{dx_1^{\,2}}+$$
$$+\frac{qh^2}{10}\left[\frac{\nu}{A}+\frac{(1-\nu^2)}{Eh}\right]+\frac{qh}{4E},$$
$$U_3^-(x_1)=W(x_1)+\frac{J}{h}(\nu-\frac{Eh}{A})\frac{d^2W}{dx_1^{\,2}}+ \quad (14)$$
$$+\frac{qh^2}{10}\left[\frac{\nu}{A}+\frac{(1-\nu^2)}{Eh}\right]-\frac{qh}{4E},$$
$$W_0(x_1)=W-\frac{EJ}{A}\frac{d^2W}{dx_1^2}+\nu\frac{qh^2}{10A},\quad \theta=-\frac{dW}{dx_1}$$

where $W(x_1)$ is the deflection function, G is the shear modulus of the beam material, q is the intensity of external load.

C. Strain Expressions

The normal and shear strains, derived within the framework of linear elasticity theory from the displacement field described by (13) and (14), are expressed as follows:

$$\varepsilon_1=-x_3\frac{d^2W}{dx_1^2},\quad \varepsilon_3=\frac{(1-\nu)}{E}\delta(x_3)q+\nu\,x_3\frac{d^2W}{dx_1^2}$$
$$\gamma_{13}=\frac{h^2 f(x_3)}{GJ}Q,\quad \gamma_{13}^0=\frac{1}{h}\int_{-z}^{z}\gamma_{13}dx_3=\theta+\frac{dW_0}{dx_1} \quad (15)$$

D. Stress Expressions

Normal bending and transverse shear stresses are determined using one-dimensional constitute laws (13) and (15) expressed as follows:

$$\sigma_1=-Ex_3\frac{d^2W}{dx_1^2}+\nu\delta(x_3)q$$
$$\sigma_3=\delta(x_3)q,\quad \tau_{13}=\frac{h^2 f(x_3)}{J}Q \quad (16)$$

E. Expression for the Cross-Sectional Bending Moment and Shear Force

The cross-sectional bending moment and shear force for a beam are defined as follows:

$$M=-EJ\frac{d^2W}{dx_1^{\,2}}+\nu\frac{qh^2}{10}$$
$$Q=-EJ\frac{d^3W}{dx_1^{\,3}}+\frac{\nu h^2}{10}\cdot\frac{dq}{dx_1} \quad (17)$$

where M and Q are the bending moment and shear force, respectively.

F. Governing Differential Equations and Boundary Conditions

By substituting the second equation of (17) into the second equation of (5), the basic equation is obtained:

$$EJ\frac{d^4W}{dx_1^4}=q+\frac{\nu h^2}{10}\frac{d^2q}{dx_1^2} \quad (18)$$

The corresponding consistent natural boundary conditions are presented in the following form:

- If the ends of the beam are hinge-supported, the boundary conditions are as follows:

$$W=0,\ M=0 \quad (19)$$

- If the ends of the beam are fixed, the boundary conditions are as follows:

$$W=0,\ \theta=0 \quad (20)$$

- If the ends of the beam are free, the boundary conditions are as follows:

$$M=0,\ Q=0 \quad (21)$$

Thus, this refined theory makes it possible to determine the stress-strain state of the beam, resolves the contradictions of the classical beam bending theory, and thereby enables calculations in a precise formulation.

The calculation of any beam using the proposed refined theory is carried out according to the following algorithm:

- The deflection function is determined by solving (18) while satisfying one of the boundary conditions specified in (19)-(21).
- The displacement components are calculated by (14).
- The strain components are determined based on (15).
- The stress components are found by (16).
- The internal forces in the beam are computed according to (17).

IV. NUMERICAL RESULTS

This section presents numerical results related to the static bending of shear-deformable isotropic prismatic rectangular beams, provided both in tabular form and as graphical representations.

The non-dimensional transverse displacement W, non-dimensional axial stress σ_1, and non-dimensional transverse shear stress τ_{13} for the beam are defined as follows:

$$\overline{W}=\frac{WEJ}{q_0 l^4},\ \overline{\sigma_1}=\frac{\sigma_1 b}{q_0},\ \overline{\tau_{13}}=\frac{\tau_{13} b}{q_0} \quad (22)$$

Numerical results for various beam thickness-to-length ratios (h/l) from Examples 1-3, calculated using the proposed theory, are presented in Tables I to VII. These findings are compared with corresponding values obtained from the two-dimensional theory of elasticity, single-variable beam theory, two-variable theory, Levinson beam theory, Timoshenko beam theory, and Bernoulli-Euler beam theory, highlighting the effectiveness of the proposed approach. In deriving the results pointed in Tables I to VII and Figures 2 to 7, several key considerations were considered:

- The Poisson's ratio μ is assumed to be 0.3.
- For Examples 1 and 2, the beam has a fixed length of l = 1 m and a width of b = 1 m. The height of the beam (h) varies and is considered for the following values: h = 0.01 m, h = 0.05 m, h = 0.10 m, and h = 0.15 m. Consequently, the h/l are 0.01, 0.05, 0.10, and 0.15. In Example 3, the beam has a fixed height of h = 1 m and a width of b = 1 m. The length of the beam (l) varies and is considered for the following values: l = 4 m and l = 10 m. Consequently, the length-to-thickness ratio ($S = l/h$) is equal to 4 and 10.
- The numerical results for $\overline{W}$ based on the Levinson beam theory and single-variable beam theory, and for $\overline{\sigma_1}$ and $\overline{\tau_{13}}$ according to the single-variable beam theory [9].
- The numerical results for $\overline{W}$, $\overline{\sigma_1}$, and $\overline{\tau_{13}}$, obtained using the two-dimensional theory of elasticity (plane stress), two-variable theory, Timoshenko beam theory, and Bernoulli-Euler beam theory, are computed by the authors [36].
- Expressions from the respective references cited in the tables are used for these calculations.
- For the Timoshenko beam theory, a shear correction factor of 5/6 is applied.

A. Example 1

A simply supported beam (SS beam) (Figure 1) subjected to a uniformly distributed transverse load. In this case, the beam ends at $x_1 = 0$ and $x_1 = l$ are simply supported. The boundary conditions for W corresponding to the SS beam are:

1) Boundary conditions at beam end $x_1 = 0$:

$$W(0)=0,\ M(0)=-EJ\frac{d^2W}{dx_1^2}+\nu\frac{qh^2}{10}\bigg|_{x_1=0}=0 \quad (23)$$

2) Boundary conditions at beam end $x_1 = l$:

$$W(l)=0,\ M(0)=-EJ\frac{d^2W}{dx_1^2}+\nu\frac{qh^2}{10}\bigg|_{x_1=l}=0 \quad (24)$$

TABLE I. NON-DIMENSIONAL TRANSVERSE DISPLACEMENT ($\overline{W}$) FOR EXAMPLE 1 (SIMPLY SUPPORTED BEAM, FIGURE 1), COMPUTED USING THE PROPOSED THEORY, WITH A COMPARISON TO EXISTING RESULTS FOR $v=0.3$

Theory	Non-dimensional transverse displacement at $x=l/2,\ \overline{W}=WEJ/(q_0 l^4)$			
	h/l= 0.01*	**h/l= 0.05***	**h/l = 0.10***	**h/l = 0.15***
Present	0.01302	0.01308	0.01329	0.01364
	(0.00 %)	(-0.15 %)	(-0.22 %)	(-0.37 %)
Bernoulli-Euler [4]	0.01302	0.01302	0.01302	0.01302
	(0.00 %)	(-0.61 %)	(-2.25 %)	(-4.89 %)
Timoshenko [4]	0.01302	0.01310	0.01335	0.01375
	(0.00 %)	(0.00 %)	(0.23 %)	(0.44 %)
Levinson [9]	0.01302	0.01310	0.01335	-
	(0.00 %)	(0.00 %)	(0.23 %)	
Single variable theory [9]	0.01302	0.01310	0.01335	-
	(0.00 %)	(0.00 %)	(0.23 %)	
Two variable theory [36]	0.01302	0.01310	0.01335	0.01375
	(0.00 %)	(0.00 %)	(0.23 %)	(0.44 %)
Theory of elasticity [4]	0.01302	0.01310	0.01332	0.01369

*Values in parentheses represent the percentage difference

TABLE II. NON-DIMENSIONAL AXIAL STRESS ($\overline{\sigma_1}$) FOR EXAMPLE 1, COMPUTED USING THE PROPOSED THEORY, WITH A COMPARISON TO EXISTING RESULTS FOR $v=0.3$

Theory	Non-dimensional axial stress at $x=l/2,\ z=h/2,\ \overline{\sigma_1}=(\sigma_1 b)/q_0$			
	h/l= 0.01*	**h/l= 0.05***	**h/l = 0.10***	**h/l = 0.15***
Present	7500.12	300.12	75.12	33.45
	(0.00 %)	(0.00 %)	(0.00 %)	(0.00 %)
Bernoulli-Euler [4]	7500.00	300.00	75.00	33.33
	(0.00 %)	(-0.07 %)	(-0.27 %)	(-0.60 %)
Timoshenko [4]	7500.00	300.00	75.00	33.33
	(0.00 %)	(-0.07 %)	(-0.27 %)	(-0.60 %)
Single variable theory [9]	7500.00	300.26	75.26	-
	(0.00 %)	(0.02 %)	(0.08 %)	
Two variable theory [36]	7500.26	300.26	75.26	33.59
	(0.00 %)	(0.02 %)	(0.08 %)	(0.18 %)
Theory of elasticity [4]	7500.20	300.20	75.20	33.53

*Values in parentheses represent the percentage difference

TABLE III. NON-DIMENSIONAL SHEAR STRESS ($\overline{\tau_{13}}$) FOR EXAMPLE 1, USING THE PROPOSED THEORY, WITH A COMPARISON TO EXISTING RESULTS FOR *v=0.3*

Theory	Non-dimensional shear stress at $x=0,\ z=h/2,\ \overline{\tau_{13}}=(\tau_{13}b)/q_0$			
	h/l= 0.01*	**h/l= 0.05***	**h/l = 0.10***	**h/l = 0.15***
Present	75.00	15.00	7.50	5.00
	(0.00 %)	(0.00 %)	(0.00 %)	(0.00 %)
Bernoulli-Euler [4]	75.00	15.00	7.50	5.00
	(0.00 %)	(0.00 %)	(0.00 %)	(0.00 %)
Timoshenko [4]	50.00	10.00	5.00	3.33
	(-33.33 %)	(-33.33 %)	(-33.33 %)	(-33.33 %)
Single variable theory [9]	75.00	15.00	7.50	-
	(0.00 %)	(0.00 %)	(0.00 %)	
Two variable theory [36]	74.92	14.92	7.42	4.92
	(-0.11 %)	(-0.53 %)	(-1.07 %)	(-1.60 %)
Theory of elasticity [4]	75.00	15.00	7.50	5.00

*Values in parentheses represent the percentage difference

B. *Example 2*

A cantilever beam (FC beam) subjected to a uniformly distributed transverse load. In this example, the beam end at $x_1 = 0$ is free, while the beam end at $x_1 = l$ is clamped. The boundary conditions for W corresponding to the FC beam are as follows:

1) Boundary Conditions at Beam End $x_1=0$:

$$M(0) = -EJ\frac{d^2W}{dx_1^{\,2}} + \nu\frac{qh^2}{10}\bigg|_{x_1=0} = 0$$
$$Q(0) = -EJ\frac{d^3W}{dx_1^{\,3}} + \frac{\nu h^2}{10}\cdot\frac{dq}{dx_1}\bigg|_{x_1=0} = 0 \tag{25}$$

2) Boundary Conditions at Beam End $x_1=l$:

$$W(l)=0, \quad \theta = -\frac{dW}{dx_1}\bigg|_{x_1=l} = 0 \tag{26}$$

TABLE IV. NON-DIMENSIONAL TRANSVERSE DISPLACEMENT ($\overline{W}$) FOR EXAMPLE 2 (CANTILEVER BEAM, FIGURE 1) USING THE PROPOSED THEORY AND COMPARED WITH EXISTING RESULTS FOR *v=0.3*

Theory	Non-dimensional transverse displacement at $x=0,\ \ \overline{W}=WEJ\ /(q_0l^4)$			
	h/l= 0.01*	**h/l= 0.05***	**h/l = 0.10***	**h/l = 0.15***
Present	0.12500	0.12545	0.12690	0.12858
	(-0.02 %)	(-0.06 %)	(-0.13 %)	(-0.82 %)
Bernoulli-Euler [2, 4]	0.12500	0.12500	0.12500	0.12500
	(-0.02 %)	(-0.41 %)	(-1.62 %)	(-3.58 %)
Timoshenko [3, 4]	0.12501	0.12533	0.12630	0.12793
	(-0.01 %)	(-0.15 %)	(-0.60 %)	(-1.32 %)
Levinson [9]	0.12502	0.12549	0.12695	-
	(0.00 %)	(-0.02 %)	(-0.09 %)	
Single variable theory [9]	0.12502	0.12549	0.12695	-
	(0.00 %)	(-0.02 %)	(-0.09 %)	
Two variable theory [36]	0.12501	0.12533	0.12630	0.12793
	(-0.01 %)	(-0.15 %)	(-0.60 %)	(-1.32 %)
Theory of elasticity [41]	0.12502	0.12552	0.12706	0.12964

*Values in parentheses represent the percentage difference

TABLE V. NON-DIMENSIONAL AXIAL STRESS ($\overline{\sigma_1}$) FOR EXAMPLE 2 DETERMINED USING THE PROPOSED THEORY AND COMPARED WITH EXISTING RESULTS FOR *v=0.3*

Theory	Non-dimensional axial stress at $x=l,\ z=h/2,\ \overline{\sigma_1}=(\sigma_1b)/q_0$			
	h/l= 0.01*	**h/l= 0.05***	**h/l = 0.10***	**h/l = 0.15***
Present	-29999.88	-1199.88	-299.88	-133.21
	(0.00 %)	(0.01 %)	(0.03 %)	(0.06 %)
Bernoulli-Euler [2, 4]	-30000.00	-1200.00	-300.00	-133.33
	(0.00 %)	(0.02 %)	(0.07 %)	(0.15 %)
Timoshenko [3, 4]	-30000.00	-1200.00	-300.00	-133.33
	(0.00 %)	(0.02 %)	(0.07 %)	(0.15 %)
Single variable theory [9]	-29999.74	-1199.74	-299.74	-
	(0.00 %)	(-0.01 %)	(-0.02 %)	
Two-variable theory [36]	-30000.00	-1200.00	-300.00	-133.33
	(0.00 %)	(0.02 %)	(0.07 %)	(0.15 %)
Theory of elasticity [41]	-29999.80	-1199.80	-299.80	-133.13

*Values in parentheses represent the percentage difference

TABLE VI. NON-DIMENSIONAL SHEAR STRESS ($\overline{\tau_{13}}$) FOR EXAMPLE 2 DETERMINED THE PROPOSED THEORY AND COMPARED WITH EXISTING RESULTS FOR *v=0.3*

Theory	Non-dimensional shear stress at $x=l,\ z=0,\ \overline{\tau_{13}}=(\tau_{13}b)/q_0$			
	h/l= 0.01*	**h/l= 0.05***	**h/l = 0.10***	**h/l = 0.15***
Present	-150.00	-30.00	-15.00	-10.00
	(0.00 %)	(0.00 %)	(0.00 %)	(0.00 %)
Bernoulli-Euler [2, 4]	-150.00	-30.00	-15.00	-10.00
	(0.00 %)	(0.00 %)	(0.00 %)	(0.00 %)
Timoshenko [3, 4]	-100.00	-20.00	-10.00	-6.67
	(-33.33 %)	(-33.33 %)	(-33.33 %)	(-33.33 %)
Single variable theory [9]	-150.00	-30.00	-15.00	-
	(0.00 %)	(0.00 %)	(0.00 %)	
Two variable theory [36]	-149.92	-29.92	14.92	-9.92
	(-0.05 %)	(-0.27 %)	(-0.53 %)	(-0.80 %)
Theory of elasticity [41]	-150.00	-30.00	-15.00	-10.00

*Values in parentheses represent the percentage difference

C. *Example 3*

TABLE I. NON-DIMENSIONAL TRANSVERSE DISPLACEMENT ($\overline{W}$) AT $x=0.25l,\ z=0$, AXIAL STRESS ($\overline{\sigma_1}$) AT $x=0.25l,\ z=h/2$ AND SHEAR STRESS ($\overline{\tau_{13}}$) AT $x=0,\ z=0$ FOR EXAMPLE 3 USING THE PROPOSED THEORY, WITH A COMPARISON TO EXISTING RESULTS FOR *v=0.3*

Source	$\overline{W}$		$\overline{\sigma_1}$		$\overline{\tau_{13}}$	
	$S=4$	$S=10$	$S=4$	$S=10$	$S=4$	$S=10$
Present	0.6865	0.5979	5.4456	32.7212	2.0000	5.0000
Bernoulli-Euler [4]	0.5811	0.5811	5.2500	32.8125	-	-
Timoshenko [4]	0.6877	0.5981	5.2500	32.8125	0.3452	0.8631
Ghugal and Sharma [17]	0.6870	0.5980	5.4406	33.0032	1.9253	4.9159
Krishna Murty [6]	0.6867	0.5980	5.4403	33.0029	1.9166	4.7917
Ghugal and Dahake [35]	0.6864	0.5979	5.4517	32.6939	1.9685	5.0646

Simply supported beam subjected to a varying load. The simply supported beam originates at the left support and is supported at $x_1 = 0$ and $x_1 = l$, with a varying load $q(x_1) = q_0(1 - \frac{x_1}{l})$ applied along its length. The boundary conditions for W are defined by (23) and (24).

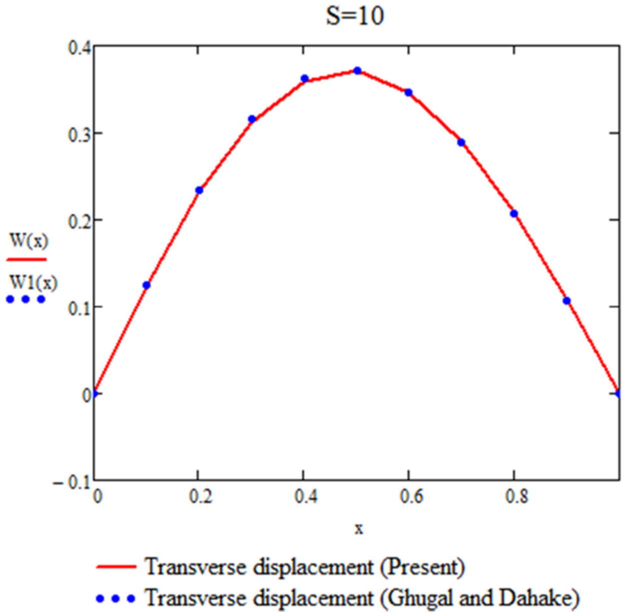

Fig. 2. Variation of transverse displacement (W) through the thickness of a simply supported beam at x = 0.25l and z, subjected to a varying load, for an aspect ratio S = 10 (Example 3).

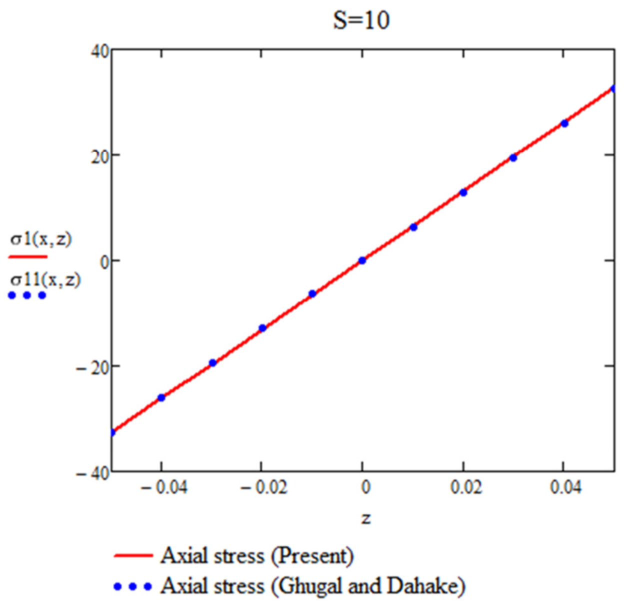

Fig. 3. Variation of axial stress (σ_1) through the thickness of a simply supported beam at x = 0.25l and z, subjected to a varying load, for an aspect ratio S = 10 (Example 3).

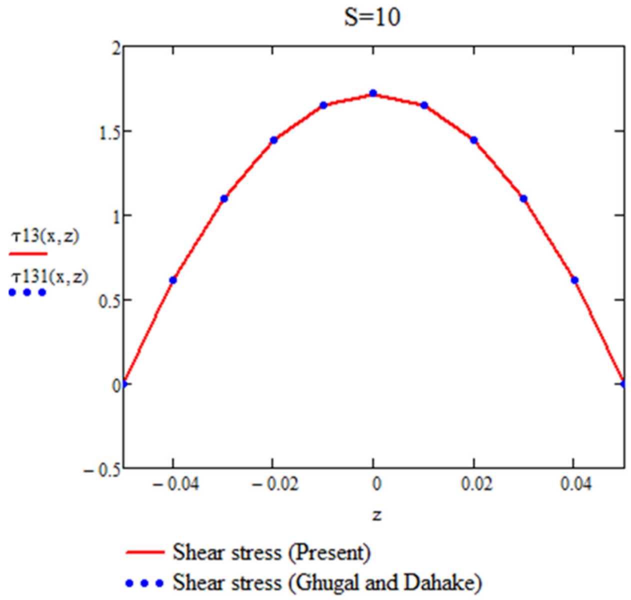

Fig. 4. Variation of shear stress (τ_{13}) through the thickness of a simply supported beam at x = 0.25l and z, subjected to a varying load, for an aspect ratio S = 10 (Example 3).

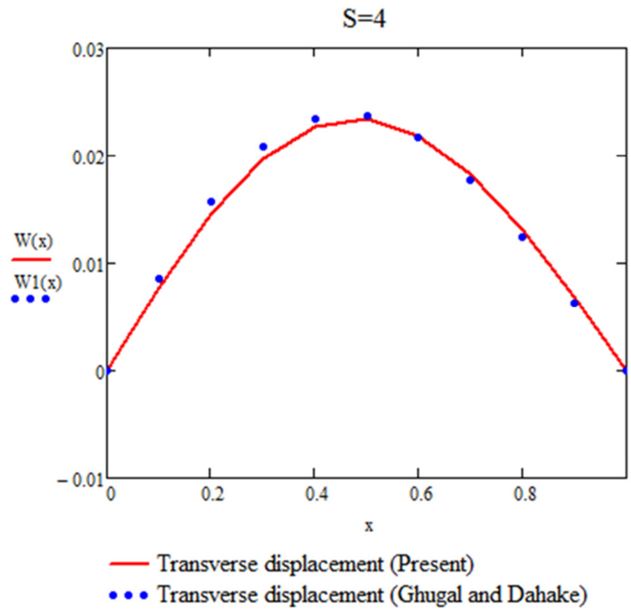

Fig. 5. Variation of transverse displacement (W) through the thickness of a simply supported beam at x = 0.25l and z, subjected to a varying load, for an aspect ratio S = 4 (Example 3).

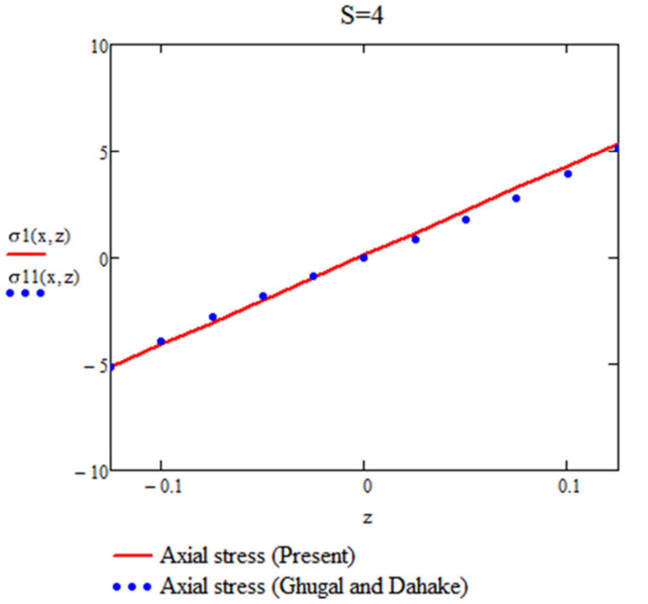

Fig. 6. Variation of axial stress (σ_1) through the thickness of a simply supported beam at x = 0.25l and z, subjected to a varying load, for an aspect ratio S = 4 (Example 3).

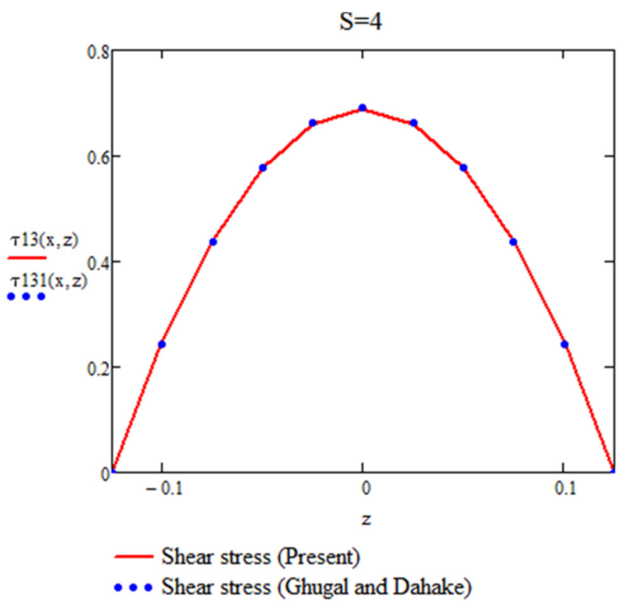

Fig. 7. Variation of shear stress (τ_{13}) through the thickness of a simply supported beam at x = 0.25l and z, subjected to a varying load, for an aspect ratio S = 4 (Example 3).

V. DISCUSSION OF NUMERICAL RESULTS FOR STATIC BEAM BENDING

This section presents a comprehensive analysis of the numerical results related to the static bending of isotropic prismatic rectangular beams, specifically in Examples 1 to 3, while considering shear deformability. Recent research on advanced beam theories has highlighted the need for improved modeling techniques that account for transverse shear deformation effects with high accuracy [42].

Tables I-VII present the findings for the dimensionless transverse deflection $\overline{W}$, the dimensionless axial stress $\overline{\sigma_1}$, and the dimensionless shear stress $\overline{\tau_{13}}$ of the beam, corresponding to SS and FC beams. Based on these results, the following observations can be detected:

- Example 1: For the (SS) beam analyzed using the present theory, the maximum percentage difference in predicting $\overline{W}$ is -0.37% (for *h/l* = 0.15), in predicting $\overline{\sigma_1}$ is 0.00% (for *h/l* = 0.15), and in predicting $\overline{\tau_{13}}$ is 0.00% (for *h/l* = 0.15). For other beam ratios (*h/l* = 0.01, *h/l* = 0.05, and *h/l* = 0.10), the percentage differences are very slight.
- Example 2: For the FC beam analyzed using the present theory, the maximum percentage difference in predicting $\overline{W}$ is -0.82% (for *h/l* = 0.15), in predicting $\overline{\sigma_1}$ is 0.06% (for *h/l* = 0.15), and in predicting $\overline{\tau_{13}}$ is 0.00% (for *h/l* = 0.15). For other beam ratios (*h/l* = 0.01, *h/l* = 0.05, and *h/l* = 0.10),, the percentage differences are minimal, indicating the robustness and accuracy of the present theory in capturing the behavior of shear-deformable beams under this boundary condition.
- Example 3: For (SS) beam subjected to a non-uniformly distributed load and analyzed using the present theory, the maximum percentage differences in $\overline{W}$, $\overline{\sigma_1}$, and $\overline{\tau_{13}}$ for *l/h* = 4 and *l/h* = 10 are negligible. These results demonstrate the accuracy and reliability of the proposed approach in predicting the response of beams under varying load conditions. For SS and FC beams, the findings obtained using the present theory align excellently with the exact solutions derived from the two-dimensional theory of elasticity, both for thin and shear-deformable beams. Furthermore, the predictions of the current theory match well with the corresponding results from the Levinson beam theory, the single-variable beam theory, and the two-variable theory for shear-deformable beams.

A key feature of the proposed theory is its ability to incorporate transverse shear deformation effects, which allows for the accurate prediction of transverse shear stresses. These stresses vary quadratically through the beam thickness and satisfy the stress-free boundary conditions at $z = \pm h/2$. This is a significant improvement over classical theories like Bernoulli-Euler and Timoshenko, which either neglect or oversimplify shear deformation effects, resulting in less precise stress predictions. Additionally, by eliminating the need for a shear correction factor, the present theory offers a more physically consistent and reliable framework for analyzing shear-deformable beams [43].

VI. CONCLUSIONS

The refined beam bending theory proposed in this study provides a significant step forward in addressing the challenges associated with classical approaches. By incorporating the effects of transverse shear deformation and eliminating the reliance on shear correction factors, this theory ensures higher accuracy and broader applicability, particularly for thick and shear-deformable beams.

Numerical results demonstrate the robustness of the proposed approach. For Simply-Supported (SS) beams, the maximum percentage difference in transverse deflection prediction is -0.37% (h/l = 0.15), while axial and shear stress predictions show negligible differences. Similarly, for fully clamped (FC) beams, the maximum percentage difference in transverse deflection is -0.82%(h/l = 0.15), with minimal deviations in axial and shear stress calculations. Furthermore, in the case of non-uniformly distributed loading on SS beams, the proposed theory accurately captures beam responses, with negligible percentage differences in transverse deflection and stresses for l/h = 4 and l/h = 10. These results validate the accuracy of the theory in modeling shear-deformable beams under various boundary conditions and loading scenarios. Furthermore, comparative analyses with existing refined theories and two-dimensional elasticity solutions validate the theory's effectiveness, while numerical examples demonstrate its applicability to various beam configurations, including simply supported, clamped, and cantilever beams under both uniform and non-uniform loads. This combination of simplicity, accuracy, and practicality highlights the potential of the proposed theory to enhance structural modeling and engineering design processes.

Future work will also involve incorporating more complex boundary conditions into the framework and applying the theory to practical engineering problems, such as the structural analysis of beams in multi-layered systems, under thermal loads, nonlinear effects, and defect analysis. Additionally, beams on elastic foundations will be studied to evaluate the influence of foundation stiffness on beam behavior, providing valuable insights for engineering design. Finally, optimization techniques will be integrated into the framework to enhance the design and performance of structural elements while ensuring practical applicability in engineering projects.

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