Refined Nonlinear Estimation of Effective Flexural Rigidity in Reinforced Concrete Beams using Curvature Integration

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ABSTRACT

Deflection control in Reinforced Concrete (RC) beams is a fundamental aspect of structural engineering. Most contemporary design codes estimate deflection using the effective moment of inertia formula, which remains largely consistent across various standards. However, an alternative and more precise approach involves computing deflection through the double integration of the moment-curvature relationship along the beam's length, offering superior accuracy but requiring significantly higher computational effort. This study evaluates deflection predictions obtained through experimental testing, conventional code-based calculations, and the moment-curvature double integration method. The findings demonstrate a strong correlation between the experimental data and the results from moment-curvature integration, whereas deflection estimates based on code formulations tend to be overly conservative. Therefore a comprehensive parametric study was performed, considering key parameters such as tensile and compressive reinforcement ratios, and span-to-depth ratio. Based on the study's findings, an empirical model is proposed to determine the effective moment of inertia, offering improved accuracy in deflection predictions while maintaining computational efficiency in RC beam analysis.

Keywords-RC beam deflection; effective moment of inertia; nonlinear analysis; moment curvature

I. INTRODUCTION

Estimating deflections in reinforced concrete (RC) beams is a fundamental consideration in structural design, ensuring compliance with serviceability limit states. Most contemporary design codes [1-5] adopt a simplified approach that utilizes a constant effective moment of Inertia (I_e) for deflection calculations. In other codes [6-10], the effective moment of inertia is replaced by the equivalent curvature. While this methodology streamlines the computational process, it often results in overly conservative deflection predictions due to its inability to account for sectional variations in stiffness.

A more refined approach involves integrating the curvature along the span of the beam [11], allowing for a more accurate representation of stiffness distribution. This methodology considers variations in the moment of inertia across different sections, leading to improved deflection estimates. However, the increased computational complexity associated with curvature integration has historically limited its widespread adoption.

Several studies have evaluated deflection prediction methodologies across various international design codes. Authors in [12] revealed that the ACI 318-03 approach yielded the most conservative deflection estimates. Similarly, researchers in [13-16] conducted comprehensive studies comparing ACI code formulations, which rely on a constant average effective moment of inertia, with curvature integration methods. The findings indicated that an equivalent moment of inertia based on curvature integration yields a stiffer structural response, thereby improving deflection prediction accuracy.

The current study seeks to address the limitations of existing methodologies by developing an empirical formula for estimating an equivalent moment of inertia based on curvature integration. This approach aims to balance computational efficiency with predictive accuracy, thereby facilitating the practical implementation of curvature-based deflection estimation in structural design. To validate the proposed model, deflection results obtained through curvature integration are systematically compared to those derived from ACI code equations and experimental data, ensuring a rigorous assessment of its applicability and reliability. The results demonstrate that the proposed approach reduces the Mean Absolute Percentage Error (MAPE) to 10.80%, compared to 46.09% and 55.62% for the ACI 318-14 and ACI 318-19 equations, respectively. Furthermore, the proposed method reduces computational time by approximately 80% compared to the curvature integration method, making it a practical tool for engineers. This improvement in accuracy and efficiency highlights the potential of the proposed approach to enhance structural design by reducing the conservatism of current code-based methods while maintaining safety and serviceability.

II. EFFECTIVE MOMENT OF INERTIA IN DESIGN CODES

Most current codes (old versions of ACI-318 up to 2014, CSA A23.3-14, and SBC-304) recommend the Branson formula for the effective moment of inertia I_e in the form (1):

$$I_e = I_{cr} + \left(\frac{M_{cr}}{M_a}\right)^3 \cdot \left(I_g - I_{cr}\right) \quad (1)$$

where I_{cr} is the moment of inertia of transformed cracked section, I_g is the gross moment of inertia of transformed un-cracked section (modular ratio method), M_{cr} is the cracking bending moment, and M_a is the applied bending moment.

In ACI-318-19, this equation has been replaced by the following Bischoff formula (2):

$$I_e = \frac{I_{cr}}{1 - \left(\frac{\frac{2}{3} M_{cr}}{M_a}\right)^2 \left(1 - \frac{I_{cr}}{I_g}\right)} \le I_g \quad (2)$$

The difference between the results of these two equations is explained in [16].

III. CURVATURE INTEGRATION METHOD

The moment of inertia can be determined using the curvature integration method, which involves the double integration of the moment-curvature relationship along the length of the beam. This approach provides a more accurate estimation of deflection by accounting for variations in curvature across the beam's span. Unlike simplified code-based methods, curvature integration considers the nonlinear behavior of reinforced concrete under loading, offering a precise representation of structural response. However, this method requires greater computational power due to its detailed analytical nature. By integrating curvature, the effective moment of inertia can be derived with enhanced accuracy, making it a valuable tool for advanced structural analysis and deflection prediction in reinforced concrete beams.

The deflection Δ is determined by double integration of curvature along the beam length in the form of (3) [17, 18]:

$$\Delta = \iint \frac{M}{EI} dx = \iint \frac{1}{R} dx = \iint \varphi dx \quad (3)$$

where M is the bending moment, E is the Young's Modulus, I is the moment of inertia, R is the radius of the curvature, and φ is the curvature.

The nonlinear effective moment I_e can then be determined by (4):

$$I_e = \frac{M_{\max} \cdot L^2}{C \cdot \Delta_{\max}} \quad (4)$$

where M_{max} is the maximum applied moment, L is the span length, C is a constant depending on loading and boundary conditions, and Δ_{max} is the maximum deflection.

IV. VERIFICATION OF CURVATURE INTEGRATION METHOD

The experimental results from [19] were utilized to validate the outcomes of the proposed nonlinear analysis method, as well as those obtained using the ACI-318-19 code approach. For this comparison, six test samples were selected according to [20]. Their properties are presented in Table I.

TABLE I. PROPERTIES OF THE SELECTED SAMPLES

Sample	section		Reinforcement		Loading	L/d
	b	h	Bottom	Top		
B1	100	300	2T16	2Y6	Third points	7.41
B2	100	300	2T16	2T10	Third points	7.41
B3	100	300	2T16	2T12	Third points	7.41
B4	100	300	2T18	2T10	Third points	7.41
B5	100	300	2T12	2Y8	Third points	7.41
B6	100	300	2T16	2T10	Mid-span	7.41

The results for beams B1, B2, and B3 are presented in Figures 1-3.

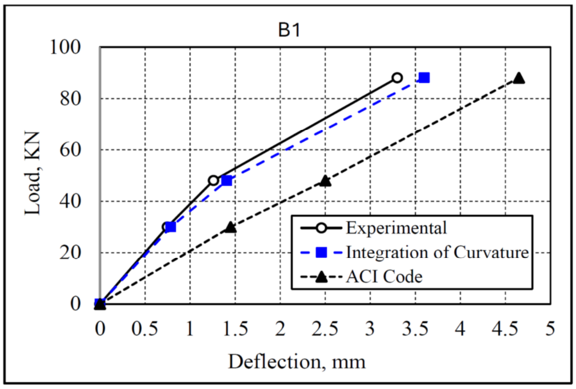

Fig. 1. Comparison of predicted deflection with experimental results, for beam B1.

The deflections calculated using the double integration of the moment-curvature curve along the beam span for the six selected models closely match the experimental results [17], whereas the (2) in ACI-318-19 code yields higher values. The loading levels considered are the crack limit, the service load limit, and at mid-point between these two limits. The curvature at these levels has been obtained using the software Response-2000 [21]. It is observed that the deflection predicted by the ACI equation is approximately 1.45 times the experimental values, while the ratio is around 0.96 for the deflections obtained through the double integration method. Additionally, the deflection ratio between the ACI equation and the proposed nonlinear analysis results is 1.5. The integration of the curvature method demonstrates strong agreement with experimental data.

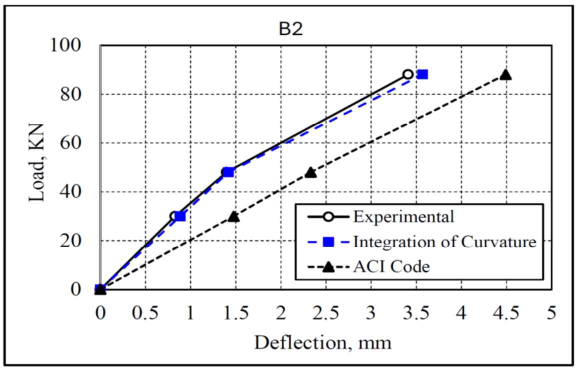

Fig. 2. Comparison of predicted deflection with experimental results, for beam B2.

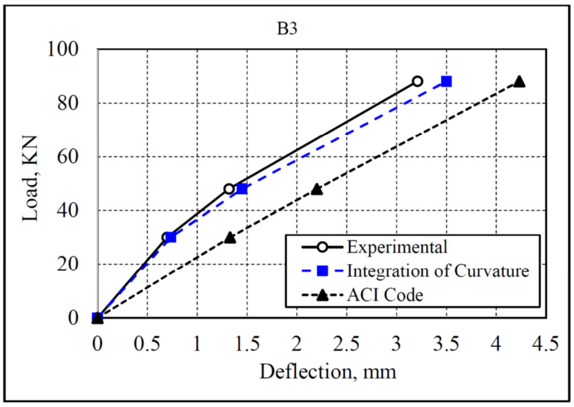

Fig. 3. Comparison of predicted deflection with experimental results, for beam B3.

V. PARAMETRIC STUDY

A comprehensive parametric study was conducted to assess the influence of key structural parameters on the deflection behavior of reinforced concrete beams. A series of simply supported beams was considered, with spans from 5 to 10 m, with span-to-depth ratios selected as 10 and 12 to evaluate the impact of slenderness. The beam cross-section width was kept constant at 200 mm for all cases to ensure consistency in geometric properties.

The tensile reinforcement percentage which is a critical factor in flexural performance, varied across four distinct levels: 0.33% (representing the minimum required for flexural resistance), 1.2%, 1.8%, and 2.1% (representing the maximum practical reinforcement for flexural applications). To further examine the effect of compression reinforcement, three different ratios were considered: 0.0 (absence of compression reinforcement), 0.25, and 0.5 of the corresponding tensile reinforcement amounts.

The material properties used in the parametric analysis were selected to reflect common structural design conditions. The concrete compressive strength was maintained at 28 MPa, while the yield strength of the reinforcing steel was set at 420 MPa. These values align with widely accepted standards and ensure the applicability of the findings to real-world structural designs. Deflection predictions for all beam configurations were computed using both the proposed nonlinear curvature integration method [22-24]. The outcomes of this parametric study contribute to the formulation of a more refined empirical model for effective moment of inertia determination, enhancing the precision and reliability of deflection predictions in reinforced concrete beam design.

The results show that deflection decreases as the tensile reinforcement percentage increases, highlighting the role of tension reinforcement in enhancing beam stiffness (Figure 4). Longer spans exhibit greater deflections due to increased bending effects. Moreover, a higher ratio of compression steel to tension steel leads to reduced deflections (Figure 5), indicating that increased compression reinforcement improves the beam's resistance to deformation. Longer spans again show higher deflections. The graphs emphasize the importance of reinforcement design in controlling deflections and consequently in the effective moment of inertia.

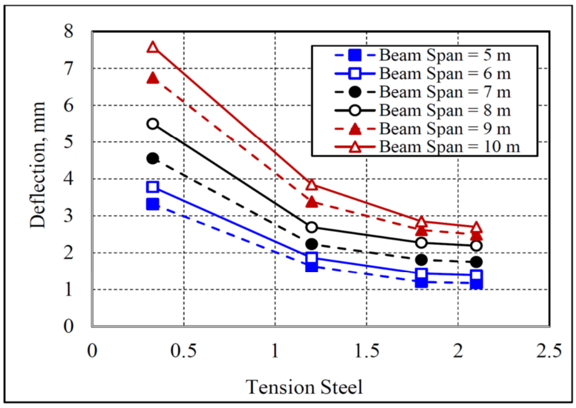

Fig. 4. Effect of tension steel reinforcement percentage on deflection.

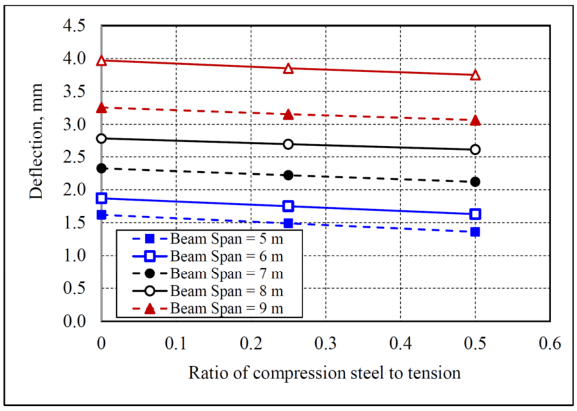

Fig. 5. Effect of compression steel reinforcement percentage on deflection.

VI. PROPOSED EQUATION

The primary dependent variable in this study is the effective moment of inertia, while the independent variables include the tensile reinforcement percentage (in the practical range from 0.25% to 2% [24]), and the compression reinforcement ratio (ranging from 0.0 to 0.5 of the tensile reinforcement percentage). Analyzing the relationship between the dependent and independent variables through scatter plots reveals nonlinearity, indicating that the effective moment of inertia increases with any rise in the independent variables. Due to the nonlinear nature of these relationships, a nonlinear regression analysis was employed to derive the best-fit equation for the effective moment of inertia. The analysis that was performed [25, 26] used a least-squares method with a very good coefficient of determination (R^2): 0.92, meaning that there is high correlation with the predicted and experimental values. The coefficients were verified for statistical significance through t-tests, yielding p-values less than 0.05 for all terms. The resulting empirical model effectively captures the influence of reinforcement ratios, providing a robust tool for predicting effective moment in reinforced concrete beams. The following equation was derived:

$$I_e = I_{cr} + 1.18 \cdot \rho^{0.57} \cdot \left(1 - 0.047 \cdot \rho'\right) \cdot \left(\frac{M_{cr}}{M_a}\right)^{0.26} \cdot \left(I_{gt} - I_{cr}\right) \quad (5)$$

where ρ is the tensile reinforcement ratio and ρ' is the compression reinforcement ratio.

Authors in [20] proposed equations for the effective moment of inertia based on the experimental results for different cases of loading considered in tests and found out that the reinforcement percentage has a significant impact in the effective moment of inertia. For the case of uniform loading the proposed formula is (6):

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^{1.43-0.457\rho} I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^{1.43-0.457\rho}\right] I_{cr} \quad (6)$$

Because (6) is based on experimental results for uniformly loaded beams, it has a set of boundaries. It does entail the percentage of reinforcement and the ratio of the cracking moment to the moment, but it does not incorporate the impact of compression reinforcement, which is one of the defining aspects of the behavior of reinforced concrete beams. On the other hand, (5) captures both the tensile and compression reinforcement ratios and also the nonlinear dependence of M_{cr}/M_a on the effective moment of inertia. The additional parameters allow (5) to accurately predict the complex deflections of reinforced concrete beams and improve the accuracy of the model in question.

The proposed equation (5) is applicable for reinforced concrete beams with concrete compressive strengths ranging from 20 MPa to 40 MPa and steel yield strengths of 420 MPa. The tensile reinforcement ratio ρ should be within the practical range of 0.25% to 2%, and the compression reinforcement ratio ρ' should not exceed 0.5 of the tensile reinforcement ratio. The equation is valid for simply supported beams with span-to-depth ratios between 10 and 12. For beams with significantly different geometries or loading conditions, such as continuous beams or beams with concentrated loads, the equation may not provide accurate results. Further research is needed to extend the applicability of the equation to these cases.

Effective moments of inertia have been calculated using (1), (2), the proposed equation (5), and (6) and their values were compared to the ones obtained from the curvature double integrations (Figure 6). Statistical analysis was carried out on the measurements and the results are presented in Table II. The proposed equation (5) produces results that align more closely with the integration of curvature compared to (1) and (2).

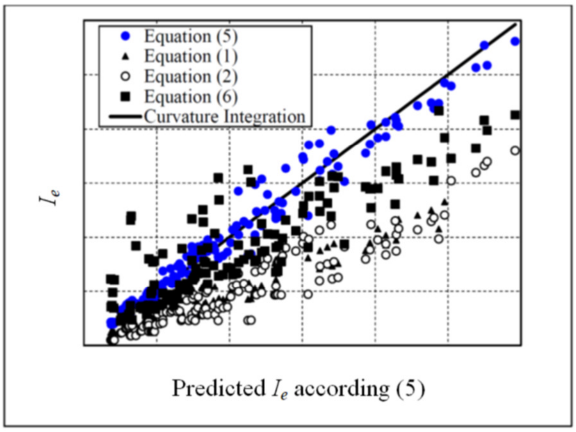

Fig. 6. Verification of the proposed formula results.

TABLE II. STATISTICAL ANALYSIS OF VARIOUS EQUATIONS RESULTS

Measure	(1)	(2)	(5)	(6)
Mean	1.89	2.43	1.00	0.96
Standard Deviation	0.25	0.72	0.14	0.54
Coefficient of variation	13.37	29.68	13.61	56.57
Mean absolute percentage error	46.09	55.62	10.80	35.13

VII. VERIFICATION OF THE PROPOSED EQUATION

To validate the accuracy of the proposed equations, the deflections obtained from experimental tests conducted in [27] were compared with the results derived using the proposed equations, as well as those calculated using the ACI-318 equations. The comparative results are summarized in Table III, which provides a detailed overview of the deflections for different models under varying load conditions. Result analysis demonstrates that the proposed equation consistently provides deflection predictions that are closer to the experimental values compared to the ACI-318 equation. Across all samples (B1 to B6), the ACI-318 predictions tend to overestimate deflections, while the proposed equation exhibits better accuracy. This trend is evident at both lower and higher load levels, reinforcing the reliability of the proposed approach in predicting structural behavior more accurately. The statistical measures presented in Table IV indicate that while the curvature integration method provides higher accuracy, it requires significant computational effort and time. In contrast, the proposed equation offers a more efficient and practical alternative, yielding results that demonstrate strong agreement with experimental data while being easier to implement.

TABLE III. VERIFICATION OF PROPOSED MODEL RESULTS

Sample	Load	Experimental	Curvature Integration	ACI	Proposal Eq. 5
B1	30	0.748	0.79	1.65	0.9
	48	1.26	1.41	2.5	1.6
	88.09	3.3	3.31	4.65	3.7
B2	30	0.829	0.885	1.476	0.95
	48	1.4	1.42	2.33	1.67
	88.09	3.41	3.57	4.49	3.8
B3	30	0.7	0.735	1.33	0.85
	48	1.32	1.45	2.2	1.56
	88.09	3.21	3.5	4.23	3.76
B4	40	0.942	1.044	1.65	0.99
	81	2.4	2.7	3.44	2.8
	122	4.05	4.44	5.23	4.7
B5	25	0.653	0.72	1.66	0.94
	40	1.43	1.60	4.12	2
	58.76	2.99	3.04	6.85	3.6
B6	15	0.645	0.562	1.03	0.72
	35	1.6	1.48	2.4	1.8
	58	3.52	3.46	3.99	3.76

TABLE IV. STATISTICAL ANALYSIS OF DEFLECTION RESULTS

Measure	Curvature Integration	ACI-318	Proposed (5)
Mean	1.05	1.76	1.19
Standard deviation	0.07	0.47	0.10
Coefficient of variation	6.71	26.75	8.45
MAPE	1.15	107.88	6.43

VIII. CONCLUSION

The current study demonstrates that the curvature integration method provides highly accurate deflection predictions for reinforced concrete (RC) beams, outperforming the ACI-318-19 Code equations. A newly developed empirical equation for estimating the effective moment of inertia offers a balance between computational efficiency and predictive accuracy, making it a practical tool for engineers.

Key factors such as tensile and compression reinforcement percentages significantly impact beam deflection, with increased reinforcement reducing deflections. Statistical analysis confirms that the proposed equation gives the lower mean absolute percentage error (10.80%). It also reduces computational time compared to the curvature integration method, providing results in a matter of seconds, making it more suitable for practical engineering applications.

The potential of the proposed method to improve structural design efficiency by reducing the conservatism of current code-based approaches while maintaining safety and serviceability is been highlighting. It also opens avenues for further research into curvature integration methods in other structural applications, enhancing accuracy and efficiency in reinforced concrete design.

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