A MTZ Formulation-Based Formulation of the Traveling Salesman Problem under Uncertainty

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ABSTRACT

The Traveling Salesman Problem (TSP) plays a key role in transport logistics, particularly in route optimization and cost reduction. In transportation and distribution, it is significant to minimize vehicle travel distance while meeting delivery deadlines. The TSP determines the shortest route that passes through multiple destinations before returning to the starting point, thereby reducing fuel consumption and travel time. This problem is the basic model for all transportation problems, and its concept is well known in operations research. This article addresses its stochastic version. The classic TSP is considered, but with a more specific case: the customers on the tour are random variables. These customers are associated with a probability pi, and the tour must be adapted to ignore absent customers. The proposed formulation is based on the original PTSP2 model. To determine the effectiveness of the proposed formulation, a comparative analysis was conducted with an existing model in the literature based on the Miller–Tucker–Zemlin (MTZ) formulation. The comparison was conducted primarily of the Continuous Relaxation Quality (QRC%) bounds determined by the two models. The two formulations were then solved via an exact solution based on a Branch-and-Cut algorithm with a constraint generation strategy that permits constraint violations to be added gradually as the optimization occurred. The models were tested with a series of test instances to evaluate their performance. The findings indicate that the proposed formulation generates substantially tighter continuous relaxation bounds. Specifically, the QRC of the proposed model has increased by over 50% compared with that of the currently existing MTZ-based model reported in the literature. Secondly, the introduced model is more computationally effective in several cases. These findings prove that the proposed formulation is effective in addressing the problem in question.

Keywords-probabilistic traveling salesman problem; constraint generation strategy; MTZ formulation; customers; continuous relaxation quality; stochastic model; two-stage formulation; computing time; classical TSP

I. INTRODUCTION

The mathematical discipline of stochastic optimization focuses on solving optimization problems in uncertain or complex contexts. In contrast to deterministic optimization, which assumes the knowledge and fixity of all problem parameters, stochastic optimization considers the unpredictable and random nature of several parameters [1]. This constitutes an effective and crucial method for multiple applications, where information is frequently partial, disturbed, or exposed to unexpected changes [2]. A stochastic optimization problem is a situation where the objective is to maximize or minimize a function (called the objective function) while considering constraints and uncertain parameters. These uncertainties are often modeled as random variables, introducing a probabilistic dimension into the problem-solving process [3]. Among the types of stochastic optimization problems, there are scenario-based problems, which allow for the design of robust solutions in uncertain environments, although they pose theoretical and practical challenges. The choice of methods depends on the type of problem, the nature of the uncertainty [4], and the operational constraints. This work focuses on a scenario-based stochastic optimization problem, known as the Probabilistic Traveling Salesman Problem (PTSP).

Optimization problems under uncertainty, such as the PTSP, play a central role in modern operations research. These problems incorporate scenarios and recourse decisions to represent the uncertainty inherent in real-world systems. The PTSP is a stochastic extension of the classical TSP. Its objective is to determine a Hamiltonian cycle that minimizes the expected tour length while accounting for possible customer cancellations. In the PTSP, an a priori (nominal) tour is first constructed according to the planned customer sequence and the probabilities of customer presence. Once the actual set of the present customers becomes known, the nominal tour is adapted by skipping absent customers, resulting in the effective tour. Although the probabilistic version of the TSP was formally studied in 2016 [5], the fundamental concept was originally introduced in [6], where the PTSP was first defined, and several important properties of optimal tours were established. Two main mathematical formulations are commonly used for the PTSP: the Dantzig–Fulkerson–Johnson (DFJ) formulation and the MTZ formulation [7]. This work focuses on the MTZ formulation, which is particularly effective for small-sized instances. The objective is to demonstrate that the MTZ formulation also performs effectively for the probabilistic version of the TSP. The MTZ formulation is one of the earliest and most widely used mathematical models for the TSP. Introduced in [8], it provides a compact representation of the problem while preventing the occurrence of subtours. Later, the probabilistic variant of the TSP was formulated in [5] using both the MTZ and DFJ approaches.

The solution methods for the PTSP can generally be classified into two main categories: heuristic approaches and exact methods. Exact methods, such as integer linear programming techniques, guarantee optimal solutions; however, their computational complexity often limits their applicability to small-sized instances. For example, the authors in [9] presented a mathematical formulation of the PTSP and applied exact solution methods to obtain numerical results for relatively small problem instances. In the present work, particular attention is given to formulations based on the MTZ model. Authors in [10] revisited the Subtour Elimination Constraints (SECs) of the MTZ formulation and proposed improvements to strengthen its performance for the Asymmetric Traveling Salesman Problem (ATSP). Furthermore, Velednitsky established in 2018 a combinatorial proof showing that the polytope defined by the DFJ formulation is contained within the polytope associated with the MTZ formulation for the ATSP. Despite this theoretical relationship, several studies have demonstrated that the MTZ formulation remains particularly effective for small-scale instances and problems of limited size [11].

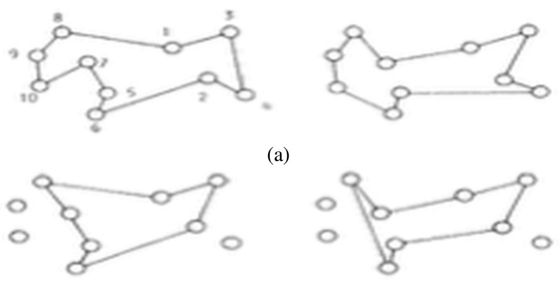
The main contributions of this paper are: A new mathematical formulation is proposed based on the PTSP2 model [8]. The formulation is presented as a two-stage stochastic programming model for the PTSP and is described in detail. A numerical comparison is then conducted between the proposed formulation and the existing PTSP1 model using randomly generated instances. The PTSP1 model is solved through an iterative algorithm that progressively adds SECs until no subtour remains in the current solution. The same iterative procedure is applied to obtain an optimal solution for the proposed PTSP model. Finally, extensive computational experiments are carried out to compare the proposed approach with models available in the literature, while the obtained results are discussed and the main conclusions of the study are presented.

II. PROBLEM DEFINITION

A. *The TSP and Scenario Approach*

The TSP is a classical problem in graph theory and combinatorial optimization in which a traveler must visit a set of cities exactly once, minimize the total travel distance, and return to the starting city [12]. The PTSP is a stochastic extension of the TSP in which each customer or node is visited with a certain probability [13]. The PTSP was first introduced in [6] as a variant of the classical TSP where only a subset of the nodes may actually be present in a given realization of the problem. In this context, an a priori tour is determined before the realization of uncertainty, and the effective tour is obtained by visiting only the customers that are present while preserving the order defined in the original tour. The objective is therefore to identify an a priori tour that minimizes the expected travel distance. The classical TSP can be viewed as a particular case of the PTSP in which all nodes are present with probability equal to one. Consequently, the main difference between the TSP and PTSP lies in the probabilistic nature of customer presence, where each node in the PTSP is associated with a visit probability ranging from 0 to 1. The relationship between the TSP and PTSP is of particular interest because the deterministic TSP represents a special case of the PTSP with no uncertainty. Several theoretical properties of the PTSP have been analyzed and rare cases have been identified in which the TSP solution remains stable under uncertainty [15].

The problem is defined on a connected graph $G = (V, E)$, where $V = \{1,2, \dots, n\}$ represents the set of customers, and E denotes the set of arcs corresponding to the travel distances or costs between customers. Each customer $i \in V$ is associated with a probability p_i, representing the likelihood that the customer must be visited. The objective is to determine a minimum-cost Hamiltonian cycle while accounting for possible customer cancellations. To model the problem, two decision variables are considered: y_{ij}, which represents the nominal tour, and x_{ij}^k, which represents the effective tour associated with scenario k. Figure 1 presents two possible realizations of the problem: Scenario A, in which all customers are present and must be visited, and Scenario B, in which some customers are absent, resulting in an adapted effective tour.



(a)

(b)

Fig. 1. Example of a tour by scenarios A and B.

B. The MTZ Formulation

The classical TSP consists of determining the shortest possible tour that visits each city exactly once and returns to the starting point, minimizing the total travel cost. The MTZ model provides one of the most widely used mathematical formulations, ensuring a compact representation and preventing sub-tours through additional ordering variables.

$$Min \sum_{i \neq j} d_{ij}\, x_{ij} \quad \text{(aa)}$$

Subject to:

$$\sum_{j=1}^{n} x_{ij} = 1 \ (\forall i \in V, i \neq j) \quad \text{(bb)}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \ (\forall j \in V, j \neq i) \quad \text{(cc)}$$

$$U_i - U_j + n x_{ij} \leq n - 1,$$

$$U_j = U_i + 1 \ si \ x_{ij} = 1, i \neq j \quad \text{(dd)}$$

$$x_{ij} \in \{0;1\} \ \forall \ (i, j) \in A \quad \text{(ee)}$$

where (aa) is the objective function that minimizes the total distance, (bb) and (cc) are the assignment constraints for the nominal tour, which ensure that each customer will visit only once, (dd) is the sub-tour elimination constraints, and (ee) is the integrality constraints.

III. MATHEMATICAL FORMULATION

A. Mathematical Formulation of the PTSP with the MTZ Model PTSP1

Equation (1) is the objective function that minimizes the total distance traveled by the effective tour for all scenarios k. (2) and (3) are the assignment constraints for the nominal tour. (4) and (5) are the assignment constraints for the adapted tour. (6) and (7) are the integrality constraints on the visit rank variable for the nominal tour. (8) are the sub-tour elimination constraints for the nominal tour. (9) and (10) are the integrality constraints for the visit rank variable for the adapted tour (scenario k). (11) are the sub-tour elimination constraints for the adapted tour. (12), (13), and (14) are the integrality constraints.

$$Min \ \frac{1}{|K|} \sum_{k=1}^{|K|} \sum_{i,j \in A^k} l_{ij} x_{ij}^k \quad (1)$$

Subject to:

$$\sum_{j=1}^{n} y_{ij} \ 1 \forall i \ \in \{1, .., n\} \quad (2)$$

$$\sum_{i=1}^{n} y_{ij} = \ 1 \ \forall j \in \{1, .., n\} \quad (3)$$

$$\sum_{j \in V_k} x_{ij}^k = 1 \forall \ i \in V_k, \forall k \in \{1, \ldots, K\} \quad (4)$$

$$\sum_{i \in V_k} x_{ij}^k = 1 \forall \ j \in V_k, \forall k \in \{1, .., K\} \quad (5)$$

$$U_i \in \mathbb{Z}, U_1 = 1 \quad (6)$$

$$2 \leq U_i \leq n \ , \forall i \in V, i \neq 1 \ et \ j \neq 1 \quad (7)$$

$$U_i - U_i + n y_{ij} \leq n - 1 \ \ , \forall i \in V, i \neq 1 \ et \ j \neq 1 \quad (8)$$

$$U_1^k = 1, \forall i \in V_k \ , \forall \ k \in \{1, .. K\} \quad (9)$$

$$2 \leq U_i^k \leq n^k, \forall i \in V_k \ , i \neq 1 \ et \ j \neq 1 \quad (10)$$

$$U_i^k - U_j^k + n^k x_{ij}^k \leq n^k - 1,$$

$$\forall i \in V_k, i \neq 1 \ et \ j \neq 1 (11) \quad (11)$$

$$x_{ij}^k \geq y_{ij}, \forall (i, j) \in A_k, \forall \ k \in \{1, .. K\} \quad (12)$$

$$y_{ij} \in \{0,1\} \forall (i, j) \in A \quad (13)$$

$$x_{ij}^k \in \{0,1\} \forall (i, j) \in A_k, \forall k \in \{1, .. K\} \quad (14)$$

B. Proposed Mathematical Formulation

This PTSP2 model was first proposed in [5]. It is a formulation based on the MTZ model. Equation (15) represents the objective function that minimizes the total distance traveled of the actual tour for all scenarios. Equations (16), (17) are the assignment constraints for the nominal tour. Equations (18), (19) are the condition constraints for the visit rank U_i^k. Equation (20) is a sub-tour elimination constraint based on the MTZ model. Equations (21), (22) are the integrality constraints.

$$\text{Min} \ \{\sum_{(i,j) \in A} l_{ij} y_{ij} + \sum_{k \in K} \sum_{A_k} l_{ij} \bar{x}_{ij}^k\} \quad (15)$$

Subject to:

$$\sum_{j:(i,j) \in A} y_{ij} + \sum_{j:(i,j) \in A_k} \bar{x}_{ij}^k = 1, \forall \ i \in V, k \in K \quad (16)$$

$$\sum_{i:(i,j) \in A} y_{ij} + \sum_{i:(i,j) \in A_k} \bar{x}_{ij}^k = 1, \forall \ j \in V, k \in K \quad (17)$$

$$U_1^k = 1, \forall k \in K \quad (18)$$

$$2 \leq U_i^k \leq n \ , \forall i \in V \ , (i \neq 1), \forall k \in K \quad (19)$$

$$U_i^k - U_j^k + n(y_{ij} + \bar{x}_{ij}^k) \leq n - 1,$$

$$\forall (i, j) \in A, (i, j \neq 1) \quad (20)$$

$$y_{ij} \in \{0,1\} \ , \forall \ (i, j) \in A \quad (21)$$

$$\bar{x}_{ij}^k \in \{0,1\}, \forall (i, j) \in A_k, k \in K \quad (22)$$

IV. SOLVING THE PROBABILISTIC TRAVELING SALESMAN PROBLEM

Since PTSP solution techniques can model a wide range of real-world applications, this problem has become an active area of research [14]. Numerous solution approaches have been proposed, including both exact and heuristic methods. In this work, an exact approach based on the professional solver CPLEX is adopted to solve the binary linear programming formulation by exploiting the relationship between the PTSP and the classical TSP [15]. In addition, a constraint generation strategy [16], like the approach proposed in [17], is employed to accelerate the solution process. The problem is solved using a Branch-and-Cut algorithm [18], as described in Algorithm 1.

A. Solving the ATSP

The ATSP is a variant of the classical TSP in which the travel cost or distance between two locations is not necessarily the same in both directions [19]. This asymmetry may result from factors such as road conditions, one-way streets, or differences in transportation and logistical costs [20]. The ATSP has important applications in logistics, transportation management, and route optimization [21]. The ATSP is classified as an NP-hard problem, meaning that obtaining an optimal solution becomes computationally difficult as the number of cities increases [22]. Classical solution approaches can generally be divided into exact and heuristic methods [23]. Exact approaches, including integer linear programming and Branch-and-Bound or Branch-and-Cut algorithms, guarantee optimal solutions; however, their computational complexity often makes them impractical for large-scale instances [24]. One of the most widely used Mixed Integer Linear Programming (MILP) formulations for the TSP is the MTZ formulation [25]. A significant challenge of this formulation is preventing the occurrence of subtours, which are cycles that do not include all nodes of the problem [26]. To address this issue, SECs are dynamically generated and added during the optimization process. In this study, the problem is solved using a Branch-and-Cut algorithm [27]. This method combines the Branch-and-Bound procedure with cutting planes to strengthen the linear relaxation and reduce the search space. At each iteration, the algorithm explores the search tree, branches on decision variables, and adds valid inequalities (cuts) whenever subtours are detected. The dynamic generation of constraints significantly improves computational efficiency, particularly for large-scale TSP instances [28].

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Algorithm 1: procedure for generating sub-
tour elimination constraints (CES)
1.initialize the problem with linear
relaxation.
2. as long as (the optimal solution is not
found):
a. solve the linear relaxation.
b. if the solution is complete and valid:
- update the best solution.
- finish if all branches are explored.
c. otherwise:
- detect sub tours.
- add valid cuts.
- if necessary, perform a branching.
3. return the best integer solution found.

Procedure Branch-and-Cut
1. Model Initialization
Define Asymmetric Traveling Salesman
Problem (ATSP) as a MILP.
using the MTZ constraints.
2. Relaxation
The next step is to relax the integrality
constraints and solve the linear
relaxation.
3. As the optimum integer solution is not
located do.
3.1 Subtour Detection
Search the existing solution and identify
subtours.
3.2 If subtours are found then
Create the relevant Subtour Elimination
Constraints (SECs).
Add the SECs to the model.
End If
3.3 In case the solution is a fraction
then
Implement the Branch-and-Bound algorithm.
Branch on the choice variables of interest
to generate subproblems.
End If
3.4 Resolve the updated model.
4. End While
5. Give back the optimal integer solution
to the optimum ATSP tour.
End Procedure
```

V. NUMERIC RESULTS

A. Comparison Between PTSP1 and PTSP2

The trials were conducted on an Intel® Core(TM) i5-4210U CPU running at 1.7 GHz with 8 GB of memory to determine the PTSP1 resolution. The programs were created in the OPL language, the modeler language of the CPLEX Solver, using 10 instances of various sizes (5, 7, and 15 nodes) and randomly generated scenario matrices. For the scenarios, random variables n_i^k were used that represent the confirmation of customers (see section nomenclature). IBM ILOG CPLEX version 12.6.0 provided the solution, as shown in Table I. The comparison of the two mathematical models for the PTSP was carried out with respect to the quality of their continuous relaxations. Figures 2 and 3 illustrate the distribution of integrality gaps, while Figure 4 reports the lower bounds obtained from the relaxations. Figures 5 and 6 further provide insights into the computational behavior associated with these relaxations.

For N = 4, it is not possible to generate 25 or 30 distinct scenarios. Therefore, all possible scenarios associated with the presence probability p(k) were considered, and the exact calculation was performed. In this case, V_{opt} represents the exact optimal value. Figure 2 displays the evolution of QRC1 and QRC2 as a function of N. QRC1, represented by the dotted line, exhibits a steady and continuous increase, whereas QRC2, represented by the solid line, increases up to N = 14 before demonstrating a significant decline. Although both models initially produce similar QRC% values, QRC1 consistently outperforms QRC2 as N increases, indicating superior relaxation quality for larger problem sizes.

TABLE I. VOPT, R, T, NDS, AND QRC OF PTSP1, PTSP2

N	|k|	V1opt	R1	N1ds	T1	V2opt	R2	N2ds	T2	QRc1%	QRc2%
4	5	130.6	129.1	0	0.02	130.6	129.6	0	0.01	1.14	0.76
6	5	144.4	141.3	25	0.01	144.4	143.5	45	0.01	2.14	0.62
8	5	139.2	135.2	0	0.03	139.2	134.3	24	0.02	2.87	3.52
10	5	147.21	140.5	30	0.22	147.21	140.7	60	0.10	4.55	4.42
12	5	134.6	125.8	25	1.2	134.6	127.4	45	0.15	6.53	5.34
14	5	146.8	132.3	140	4.43	146.8	136.7	180	0.6	9.87	6.88
15	5	148.3	131	720	8.60	148.3	140.6	1256	1.1	11.66	5.19
4	10	110.8	105.2	0	0.07	110.8	100.6	0	0.02	5.05	9.2
6	10	118.7	102.4	30	0.03	118.7	107.2	47	0.01	13.73	9.68
8	10	123.6	113.6	120	0.1	123.6	114.3	157	0.1	8.09	7.52
10	10	145.3	133.2	432	0.5	145.3	133.7	674	0.3	8.32	7.98
12	10	138.7	123.2	42	4	138.7	129.4	78	2.12	11.17	6.7
14	10	141.8	126.1	18432	6.61	141.8	125.7	872	3.13	11.07	11.35
15	10	145.2	125.9	7654	8.12	145.2	137.1	9750	36.05	13.29	5.57
4	20	132.5	122.7	0	0.02	132.5	126.4	0	0.03	7.39	4.6
6	20	110.8	100.3	6	0.03	110.8	97.6	12	0.05	9.47	11.91
8	20	130.8	121.8	12	0.25	130.8	120.3	24	0.26	6.88	8.02
10	20	143.2	130.2	412	3.4	143.2	130.4	523	1.4	9.07	8.93
12	20	139.8	123.7	2312	5.8	139.8	124.5	2800	5.77	11.51	10.94
14	20	140.9	120.7	765	12.8	140.9	115.7	1254	1860	14.33	17.88
15	20	144.8	124.7	413	20.9	144.8	116.7	834	2400	13.88	19.4
4	25	131.9	120.9	0	0.02	131.9	124.8	0	0.02	8.33	5.38
6	25	109.1	99.1	8	0.07	109.1	97.7	23	0.06	9.16	10.44
8	25	130.3	118.2	14	0.5	130.3	120.4	33	0.55	9.28	7.59
10	25	150.4	130.2	400	2.2	150.4	138.2	754	1.75	13.43	8.11
12	25	134.1	110.2	1230	55.12	134.1	113.5	2953	9.83	17.82	15.36
14	25	140.6	122.5	900	60.8	140.6	130.5	975	2160.7	12.87	7.18
15	25	142.8	122	510	78.9	142.8	125	812	2900.3	14.56	12.46

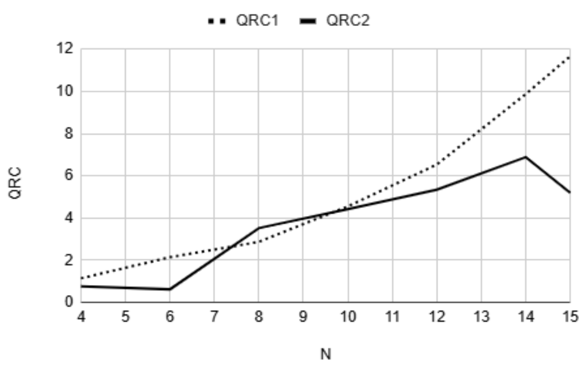

Fig. 2. QRC1 and QRC2 for 5 scenarios.

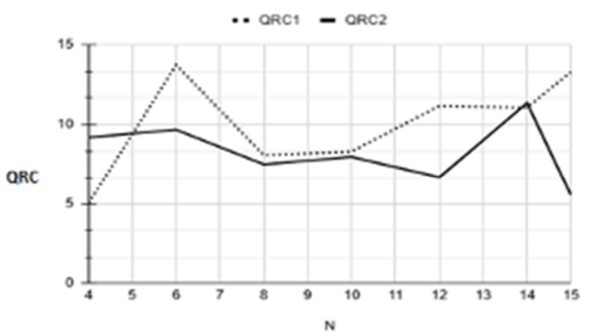

Fig. 3. QRC1 and QRC2 for 10 scenarios.

Figure 3 depicts the values of N ranging from 4 to 15; the graph displays the QRC for two distinct approaches, QRC1 (dotted line) and QRC2 (solid line), over 10 distinct circumstances. QRC1 is less stable, but it can occasionally offer better relaxation quality. QRC2 is more reliable and consistent, which may be better in real-world situations where stability is essential. The decision between QRC1 and QRC2 most likely comes down to whether more stable outcomes (QRC2) or higher average relaxation quality (QRC1) are desired.

In Figure 4, it is observed that for small N, both methods perform similarly, whereas for large N, QRC2 is superior in terms of relaxation quality.

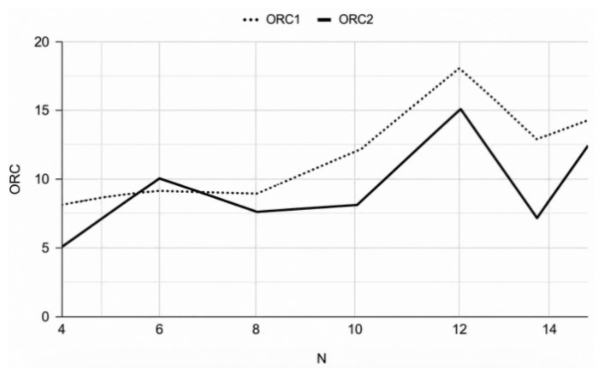

Fig. 4. QRC1 and QRC2 for 20 scenarios.

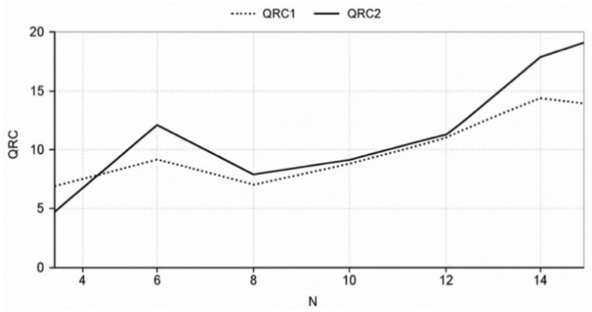

Fig. 5. QRC1 and QRC2 for 25 scenarios.

Figure 5 demonstrates that, on average, QRC1 offers higher relaxation quality, although there are notable variations. Because QRC2 is steadier, it may be more dependable in real-world applications. QRC1 exhibits stronger but unstable results for bigger N (N ≥ 10), while QRC2 has a more consistent growth trend. Whether stability (QRC2) or higher peak performance (QRC1) is desired may influence the decision between QRC1 and QRC2.

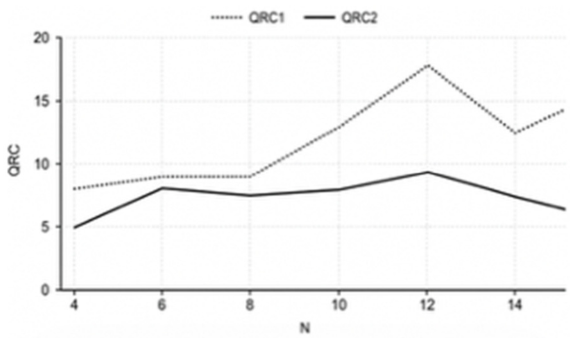

Fig. 6. QRC1 and QRC2 for 30 scenarios.

Figure 6 illustrates the QRC values obtained for the two approaches, QRC1 and QRC2, for different values of N ranging from 4 to 15 under 30 scenarios. The observed differences between QRC1 and QRC2 indicate that the two approaches produce different relaxation quality performances. QRC1 exhibits greater variability but shows improved relaxation quality as N increases, whereas QRC2 provides more stable results across the tested instances.

B. Analysis of the Solutions

Several important conclusions can be drawn from the analysis of the figures comparing QRC1 and QRC2. The Quality of Continuous Relaxation (QRC) is a metric used to evaluate the effectiveness of the linear relaxation of an integer programming formulation for the PTSP. It is defined as the ratio between the objective value obtained from the relaxed (continuous) model and the objective value of the best known feasible integer solution. A QRC value closer to 1 indicates a tighter relaxation and a better approximation of the optimal solution, whereas lower values correspond to weaker relaxations and looser bounds. Based on the numerical results and graphical analysis, the PTSP1 model, which is based on the classical MTZ formulation, generally provides better performance than PTSP2 for small and medium-sized instances. However, as the number of nodes N increases, PTSP2 appears to become more competitive and may eventually outperform PTSP1 for larger instances. This observation should be further investigated through experiments on larger-scale problems. The comparative analysis of QRC1 and QRC2 also reveals a trade-off between relaxation quality and stability. QRC1 often achieves better relaxation quality, but its performance is more sensitive to variations in problem parameters. In contrast, QRC2 provides more stable and consistent results, although its relaxation quality is generally lower. Therefore, the choice between the two approaches should depend on the objectives of the optimization process and the specific characteristics of the problem being addressed.

VI. CONCLUSIONS

This study investigated the Probabilistic Traveling Salesman Problem (PTSP) using exact solution methods based on the MTZ formulation. More specifically, a new two-stage stochastic programming formulation incorporating MTZ-type constraints was proposed, and a comparative analysis with the existing formulations from the literature was conducted. The computational results demonstrated the effectiveness of the proposed approach, particularly in terms of improving the quality of the linear relaxation and reducing the number of Branch-and-Cut nodes required during the optimization process. The results also confirmed that the PTSP can be solved exactly for small-sized instances, allowing efficient routing decisions to be identified while explicitly accounting for uncertainty in customer presence. Among the formulations considered, PTSP1 generally outperformed PTSP2 for small-scale instances. Furthermore, the proposed approach remained relatively simple to implement and required a moderate number of variables and constraints. Despite these promising results, the scalability of the proposed formulations remains a significant limitation. Exact methods become computationally expensive for instances involving more than approximately 40 customers. Consequently, future research should focus on extending the proposed approach to larger-scale PTSP instances. Potential research directions include the development of advanced decomposition techniques, parallel computing strategies, and hybrid approaches combining exact optimization methods with customized metaheuristics. Such developments could significantly improve computational efficiency and enhance the applicability of the proposed models to real-world routing problems characterized by both large-scale structures and uncertainty.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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DATA AVAILABILITY

The data used to support the findings of this study are available from the corresponding author.

REFERENCES

[1] D. Fouskakis and D. Draper, "Comparing Stochastic Optimization Methods for Variable Selection in Binary Outcome Prediction, With Application to Health Policy," *Journal of the American Statistical Association*, vol. 103, no. 484, pp. 1367–1381, Dec. 2008, https://doi.org/10.1198/016214508000001048.

[2] M.-H. Lin, J.-F. Tsai, and C.-S. Yu, "A Review of Deterministic Optimization Methods in Engineering and Management," *Mathematical Problems in Engineering*, vol. 2012, no. 1, 2012, Art. no. 756023, https://doi.org/10.1155/2012/756023.

[3] J. C. Spall, "Stochastic Optimization," in *Handbook of Computational Statistics: Concepts and Methods*, J. E. Gentle, W. K. Härdle, and Y. Mori, Eds. Berlin, Heidelberg: Springer, 2012, pp. 173–201.

[4] L. A. Hannah, "Stochastic optimization," *International Encyclopedia of the Social & Behavioral Sciences*, vol. 2, pp. 473–481, 2015.

[5] P. Adasme, R. Andrade, J. Leung, and A. Lisser, "A Two-stage Stochastic Programming Approach for the Traveling Salesman Problem," presented at the International Conference on Operations Research and Enterprise Systems, Feb. 2016, vol. 2, pp. 163–169, https://doi.org/10.5220/0005738801630169.

[6] P. Jaillert, "Probabilistic Traveling Salesman Problems," M.S. Thesis, Massachusetts Institute of Technology, USA, 1985.

[7] L. Zhou, Y. Ju, and S. Moura, "Learning-Guided Local Search for Asymmetric Traveling Salesman Problem," presented at the 2nd AI for Math Workshop @ ICML 2025, July 2025, [Online]. Available: https://openreview.net/forum?id=PlEn4KjSTN&noteId=PlEn4KjSTN.

[8] C. E. Miller, A. W. Tucker, and R. A. Zemlin, "Integer Programming Formulation of Traveling Salesman Problems," *J. ACM*, vol. 7, no. 4, pp. 326–329, July 1960, https://doi.org/10.1145/321043.321046.

[9] E. dos S. Teixeira and S. A. de Araujo, "Formulations for the Traveling Salesman Problem with D-Relaxed Priority Rule." Social Science Research Network, Rochester, NY, Aug. 14, 2022, https://doi.org/10.2139/ssrn.4189790.

[10] T. Bektaş and L. Gouveia, "*Requiem* for the Miller–Tucker–Zemlin subtour elimination constraints?," *European Journal of Operational Research*, vol. 236, no. 3, pp. 820–832, Aug. 2014, https://doi.org/10.1016/j.ejor.2013.07.038.

[11] M. Velednitsky, "Short Combinatorial Proof that the DFJ Polytope is contained in the MTZ Polytope for the Asymmetric Traveling Salesman Problem," *Operations Research Letters*, vol. 45, no. 4, pp. 323–324, July 2017, https://doi.org/10.1016/j.orl.2017.04.010.

[12] G. Dantzig, R. Fulkerson, and S. Johnson, "Solution of a Large-Scale Traveling-Salesman Problem," *Journal of the Operations Research Society of America*, vol. 2, no. 4, pp. 393–410, Nov. 1954, https://doi.org/10.1287/opre.2.4.393.

[13] P. L. J. Wissink, "The Traveling Salesman Problem with Stochastic and Correlated Customers," *Transportation Science*, vol. 57, no. 5, pp. 1321–1339, Sept. 2023, https://doi.org/10.1287/trsc.2022.0005.

[14] Ф. Ель Асрі, К. Таяні, Х. Фахурі, F. El Asri, C. Tajani, and H. Fakhouri, "A combined ant colony optimization with Levy flight mechanism for the probabilistic traveling salesman problem with deadlines," Feb. 2024, https://doi.org/10.23939/mmc2024.01.290.

[15] M. Blanchard, A. Jacquillat, and P. Jaillet, "Probabilistic Bounds on the k-Traveling Salesman Problem and the Traveling Repairman Problem," *Mathematics of Operations Research*, vol. 49, no. 2, pp. 1169–1191, May 2024, https://doi.org/10.1287/moor.2021.0286.

[16] R. Salman, F. Ekstedt, and P. Damaschke, "Branch-and-bound for the Precedence Constrained Generalized Traveling Salesman Problem," *Operations Research Letters*, vol. 48, no. 2, pp. 163–166, Mar. 2020, https://doi.org/10.1016/j.orl.2020.01.009.

[17] T. Q. T. Vo, M. Baiou, and V. H. Nguyen, "A branch-and-cut algorithm for the balanced traveling salesman problem," *Journal of Combinatorial Optimization*, vol. 47, no. 2, Jan. 2024, Art. no. 4, https://doi.org/10.1007/s10878-023-01097-4.

[18] R. Tadei, G. Perboli, and F. Perfetti, "The multi-path Traveling Salesman Problem with stochastic travel costs," *EURO Journal on Transportation and Logistics*, vol. 6, no. 1, pp. 3–23, Mar. 2017, https://doi.org/10.1007/s13676-014-0056-2.

[19] S. Rosenow, "A heuristic for the probabilistic traveling salesman problem," in *Operations Research Proceedings 1996*, 1997, pp. 117–122, https://doi.org/10.1007/978-3-642-60744-8_22.

[20] G. Campuzano, C. Obreque, and M. M. Aguayo, "Accelerating the Miller–Tucker–Zemlin model for the asymmetric traveling salesman problem," *Expert Systems with Applications*, vol. 148, June 2020, Art. no. 113229, https://doi.org/10.1016/j.eswa.2020.113229.

[21] É. D. Taillard, "A linearithmic heuristic for the travelling salesman problem," *European Journal of Operational Research*, vol. 297, no. 2, pp. 442–450, Mar. 2022, https://doi.org/10.1016/j.ejor.2021.05.034.

[22] G. Angulo and D. Moran, "On parametric formulations for the Asymmetric Traveling Salesman Problem." arXiv, Nov. 21, 2024, https://doi.org/10.48550/arXiv.2411.13758.

[23] M. Lopez-Ibanez and T. Stutzle, "The Automatic Design of Multiobjective Ant Colony Optimization Algorithms," *IEEE Transactions on Evolutionary Computation*, vol. 16, no. 6, pp. 861–875, Sept. 2012, https://doi.org/10.1109/TEVC.2011.2182651.

[24] R. Gan, Q. Guo, H. Chang, and Y. Yi, "Improved ant colony optimization algorithm for the traveling salesman problems," *Journal of Systems Engineering and Electronics*, vol. 21, no. 2, pp. 329–333, Apr. 2010, https://doi.org/10.3969/j.issn.1004-4132.2010.02.025.

[25] T.-L. Jin, Q.-J. Jiang, and Z.-K. Shao, "Particle Swarm Optimization-Based Algorithms for Traveling Salesman Problem," in 2025 Joint International Conference on Automation-Intelligence-Safety (ICAIS) & International Symposium on Autonomous Systems (ISAS), Feb. 2025, pp. 1–5, https://doi.org/10.1109/ICAISISAS64483.2025.11052121.

[26] C. Blum and A. Roli, "Metaheuristics in combinatorial optimization: Overview and conceptual comparison," *ACM Comput. Surv.*, vol. 35, no. 3, pp. 268–308, June 2003, https://doi.org/10.1145/937503.937505.

[27] H. Jafarzadeh, N. Moradinasab, and M. Elyasi, "An Enhanced Genetic Algorithm for the Generalized Traveling Salesman Problem," *Engineering, Technology & Applied Science Research*, vol. 7, no. 6, pp. 2260–2265, Dec. 2017, https://doi.org/10.48084/etasr.1570.

[28] M. Abdul-Niby, M. Alameen, A. Salhieh, and A. Radhi, "Improved Genetic and Simulating Annealing Algorithms to Solve the Traveling Salesman Problem Using Constraint Programming," *Engineering, Technology & Applied Science Research*, vol. 6, no. 2, pp. 927–930, Apr. 2016, https://doi.org/10.48084/etasr.627.