Stochastic Buckling Analysis of Non-Uniform Columns Using Stochastic Finite Elements with Discretization Random Field by the Point Method

Do Thi Hang
University of Transport and Communications
Hanoi, Vietnam
hangdo@utc.edu.vn

Nguyen Xuan Tung
University of Transport and Communications
Hanoi, Vietnam
ngxuantung@utc.edu.vn

Dao Ngoc Tien
Hanoi Architectural University
Hanoi, Vietnam
tiendn@hau.edu.vn

Received: 9 February 2022 | Revised: 9 March 2022 | Accepted: 10 March 2022

Abstract-This study examined the discretization random field of the elastic modulus by a point method to develop a stochastic finite element method for the stochastic buckling of a non-uniform column. The formulation of stochastic analysis of a non-uniform column was constructed using the perturbation method in conjunction with the finite element method. The spectral representation was used to generate a random field to employ the Monte Carlo simulation for validation with a stochastic finite element approach. The results of the stochastic buckling problem of non-uniform columns with the random field of elastic modulus by comparing the first-order perturbation technique were in good agreement with those obtained from the Monte Carlo simulation. The numerical results showed that the response of the coefficient of variation of critical loads increased when the ratio of the correlation distance of the random field increased.

Keywords-non-uniform column; buckling; stochastic FEM; spectral representation; random field

I. INTRODUCTION

Science and technology revolution has produced many high-strength and lightweight materials used in a variety of slender structures such as steel structures [1-7] and functionally graded beams [8]. The problem of stability calculation of these structures is very important, e.g. in a steel truss bridge with many slender compression members. In many cases, column or tower structures are designed with variable cross-sections to match the bearing characteristics of the structure and save materials. Many studies have examined the stability of bars with variable cross-section and many basic problems on the stability of bars with variable cross-section were presented in [9-12]. In addition, many researchers studied the stability of the replacement with a changing cross-section with different forms, calculated by different methods. In [13], a polynomial series approximation was used to solve the problem of column stability with variable cross-sections subjected to axial forces. In [14], the column was calculated with a cross-sectional variation of the ladder using the Rayleigh-Ritz method. In [15], the stability of a column of the variable cross-section with elastic connections was calculated by approximating the stability differential equation. In [9], the stability of reinforced concrete columns with elastic connections was studied. An exact solution for some types of columns with variable cross-sections with elastic connections under distributed axial forces was presented in [16]. In addition to analytical methods with exact solutions, some studies used finite element methods. In [17], the stability of a bar with variable cross-section was studied using the Galerkin finite element method. In [18], the stability calculation of columns with stepped and cracked cross-sections was studied using the finite element method. In [19], a finite element model of circular concrete-filled steel circular-tube columns was studied under axial compression loading.

However, these studies were limited to deterministic problems. In most practical engineering, structure analysis ignores the random heterogeneity of materials. For the advanced analysis of structures, a probabilistic model is needed to consider random variables, random fields of geometry, and material properties of structures. Several methods for random field discretization were developed for stochastic finite element methods, such as the integration point method [20, 21], the nodal point method [22, 23], and the local averaging method [24]. In [25], a stochastic finite element method was presented using the weighted integration approach to analyze the static behavior of non-uniform columns. In [26], a semi-analytical method was applied to investigate the buckling of composite beam-columns with random elastic stiffness and geometric properties. In [27], the free vibration of functionally graded beams was examined using a stochastic perturbation-based

Corresponding author: Nguyen Xuan Tung

finite element method. The random finite element method was improved for the seismic analysis of gravity dams in [28]. In [29, 30], a probabilistic problem was developed for the seismic analysis of cabinet facilities in nuclear power plants and soil properties.

In this study, a stochastic finite element model was built for the buckling of the non-uniform column with the random field of elastic modulus.

II. THE FINITE ELEMENT MODEL OF THE NON-UNIFORM COLUMN

Consider a non-uniform column with the coordinate system (x, z) as shown in Figure 1:

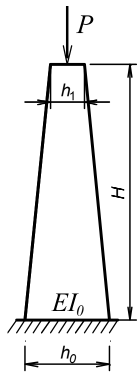

Fig. 1. Model of non-uniform columns

The column element is assumed to have two degrees of freedom, one rotation and one translation, at each end. The location and positive directions of these displacements in a typical linearly tapered column element are shown in Figure 2, where L_e is the length of the element, E and I are the area and inertia moment of the cross-section column. The depths of the cross-sections at the smaller and larger end of the column are denoted as h_2 and h_1, respectively. The longitudinal axis of the element lies along the x-axis. The element was assumed to have two degrees of freedom at each end: a transverse deflection u_1, u_3, and an angle of rotation of slope u_2, u_4.

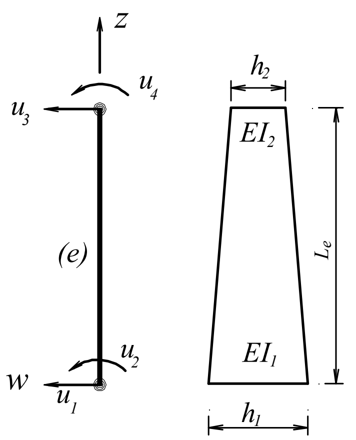

Fig. 2. Model of a non-uniform finite element.

The displacement field is approximated by the shape function as:

$$w_e = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = [N]\{u\}_e \quad (1)$$

where $N = \{N_1 \quad N_2 \quad N_3 \quad N_4\}$, and N_i is the shape function of the *i-th* degree of freedom, and the Hermite polynomial functions [31] are:

$$\begin{aligned} N_1 &= 1 - 3\frac{z^2}{L_e^2} + 2\frac{z^3}{L_e^3} \\ N_2 &= z\left(1 - 2\frac{z}{L_e} + \frac{z^2}{L_e^2}\right) \\ N_3 &= 3\frac{z^2}{L_e^2} - 2\frac{z^3}{L_e^3} \\ N_4 &= z\left(-\frac{z}{L_e} + \frac{z^2}{L_e^2}\right) \end{aligned} \quad (2)$$

The deformation potential of the column is:

$$U_e = \frac{1}{2}\int_0^{L_e} EI(z)\left(\frac{d^2 w_e}{dz^2}\right)^2 dz \quad (3)$$

The potential energy of load is:

$$V_e = -\frac{P}{2}\int_0^{L_e}\left(\frac{dw_e}{dz}\right)^2 dz \quad (4)$$

The cross-section moment of inertia in the element is approximated by linear interpolation:

$$E(z)I_e(z) = E(z)I_{1e}\left(1 - \frac{z}{L_e}\right) + E(z)I_{2e}\frac{z}{L_e} \quad (1)$$

The random field of elastic modulus is assumed as:

$$E(z) = E_0[1 + r(z)] \quad (2)$$

where E_0, $r(z)$ are the mean elastic modulus and a one-dimensional Gaussian random field with a mean equal to zero respectively. The form of the autocorrelation function of random field $r(x)$ is:

$$R(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(z)r(z+\tau)f_z(z, z+\tau, \tau)dr(z)dr(z+\tau) \quad (7)$$

The random field elastic modulus requires discretization to random variables for the governing equation of buckling problem by finite element formulation.

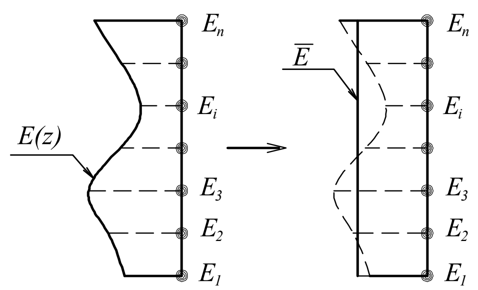

Fig. 3. Average model for approximating random field of elastic modulus.

By averaging random variables within the element as illustrated in Figure 3, the random field of the elastic modulus in the element is calculated as:

$$\bar{r} = \frac{r_1 + r_2 + \cdots + r_n}{n}$$
$$\overline{E} = E_0\left[1 + \frac{r_1 + r_2 + \cdots + r_n}{n}\right] \quad (8)$$

Taking the variation with respect to q, the resulting equation [32] is obtained for the buckling of the column:

$$\left([K] - \lambda [K]_g\right)\{u\} = 0 \quad (9)$$

[K] and [M] denote the assembled and global stiffness respectively of the cross-section column and λ_i denotes the critical load. The stiffness matrix $[K]_e$ is defined as:

$$[K]_e = E_0\left[1 + \frac{r_1 + r_2 + \cdots + r_n}{n}\right] \times$$
$$\times \begin{bmatrix} \frac{6(I_{1e}+I_{2e})}{L_e^3} & \frac{4I_{1e}+2I_{2e}}{L_e^2} & -\frac{6(I_{1e}+I_{2e})}{L_e^3} & \frac{2I_{1e}+4I_{2e}}{L_e^2} \\ & \frac{3I_{1e}+I_{2e}}{L_e} & -\frac{4I_{1e}+2I_{2e}}{L_e^2} & \frac{I_{1e}+I_{2e}}{L_e} \\ & & \frac{6(I_{1e}+I_{2e})}{L_e^3} & -\frac{2I_{1e}+4I_{2e}}{L_e^2} \\ Sym. & & & \frac{I_{1e}+3I_{2e}}{L_e} \end{bmatrix} \quad (10)$$

The geometric stiffness matrix is defined as:

$$[K]_{ge} = P\begin{bmatrix} \frac{6}{5L_e} & \frac{1}{10} & -\frac{6}{5L_e} & \frac{1}{10} \\ & \frac{2}{15L_e} & -\frac{1}{10} & -\frac{1}{30L_e} \\ & & \frac{6}{5L_e} & -\frac{1}{10} \\ \text{Sym} & & & \frac{2}{15L_e} \end{bmatrix} \quad (3)$$

III. FORMULATION OF THE STOCHASTIC FINITE ELEMENT METHOD USING THE PERTURBATION TECHNIQUE FOR THE BUCKLING OF THE COLUMN

The governing equation of the buckling problem in (9) includes random variables and can be perturbed concerning the mean of the random variables as follows:

$$\begin{pmatrix} [K]_0 + \sum_{i=1}^{Nr}\frac{\partial[K]}{\partial r_i}r_i - \\ \left(\lambda_0 + \sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_i}r_i\right)[K]_g \end{pmatrix}\left\{u_0 + \sum_{i=1}^{Nr}\frac{\partial u}{\partial r_i}r_i\right\} = 0 \quad (12)$$

Solving the stochastic equation (12), the zeroth-order and the first-order solutions can be obtained as:

zeroth-order:

$$\left([K]_0 - \lambda_0[K]_g\right)\{u_0\} = 0 \quad (4)$$

and first-order:

$$\left([K]_0 - \lambda_0[K]_g\right)\left\{\frac{\partial u}{\partial r_i}\right\} = -\left(\frac{\partial[K]}{\partial r_i} - \frac{\partial\lambda}{\partial r_i}[K]_g\right)\{u_0\} \quad (14)$$

Premultiplication of (14) by $\left\{u_0 + \sum_{i=1}^{Nr}\frac{\partial u}{\partial r_i}r_i\right\}^T$ gives:

$$\left\{u_0 + \sum_{i=1}^{Nr}\frac{\partial u}{\partial r_i}r_i\right\}^T \times$$
$$\times\left\{[K]_0 + \sum_{i=1}^{Nr}\frac{\partial[K]}{\partial r_i}r_i - \left(\lambda_0 + \sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_i}r_i\right)[K]_g\right\} \quad (15)$$
$$\times\left\{u_0 + \sum_{i=1}^{Nr}\frac{\partial u}{\partial r_i}r_i\right\} = 0$$

Solving (15) using the orthonormal property, the first-order partial derivatives of critical load respect random variable is given by:

$$\frac{\partial\lambda}{\partial r_i} \approx \frac{\{u_0\}_i^T\frac{\partial[K]}{\partial r_i}\{u_0\}}{\{u_0\}_i^T[M]\{u_0\}} \quad (16)$$

The mean of the critical load given solved by the first-order perturbation solutions is:

$$\mu_\lambda = \int_{-\infty}^{\infty}\left\{\left(\lambda_0 + \sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_i}r_i\right) - \lambda_0\right\}p(r_i)dr_i = \lambda_0 \quad (17)$$

The variance of the critical load is solved by the first-order perturbation solution as follows:

$$Var_\lambda = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{Bmatrix}\left\{\left(\lambda_0 + \sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_i}r_i\right) - \lambda_0\right\}\times \\ \times\left(\lambda_0 + \sum_{j=1}^{Nr}\frac{\partial\lambda}{\partial r_j}r_j\right) - \lambda_0\end{Bmatrix}\times \quad (18)$$
$$\times f_z(r_i, r_j, r_j - r_i)dr_i dr_j$$
$$= \sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\left\{\frac{\partial\lambda}{\partial r_i}\frac{\partial\lambda}{\partial r_i}R(\tau)\right\}$$

where $R(\tau)$ denotes the autocorrelation function of the random field, and the relative distance vector is defined as $\tau = r_j - r_i$. The autocorrelation function was assumed in the form:

$$R(\tau) = \sigma^2\, exp\left(-\frac{\tau^2}{d^2}\right) \quad (19)$$

where σ, d are the Coefficient Of Variation (COV) and the correlation distance of the random field of elastic modulus respectively. The response variability can be represented using the COV defined as:

$$COV = \frac{\sqrt{Var_\lambda}}{|\mu_\lambda|} \quad (20)$$

IV. EXAMPLES

A. Validation of the Finite Element Approach for Deterministic Analysis

The buckling non-uniform column in Figure 1 was considered to validate the proposed finite element for the non-uniform column with the moment of inertia as a formulation:

$$EI(z) = \frac{EI_0}{2}\left\{\sqrt{2} - \frac{x}{H}\left(\sqrt{2} - 1\right)\right\}^2 \quad (21)$$

The analytical solutions of the column were given by [9] with a critical load factor m=2.023:

$$P_{cr} = 2.023\frac{EI_0}{H^2} \quad (22)$$

where the critical load factor m is given by:

$$m = \frac{P_{cr}H^2}{EI_0} \quad (5)$$

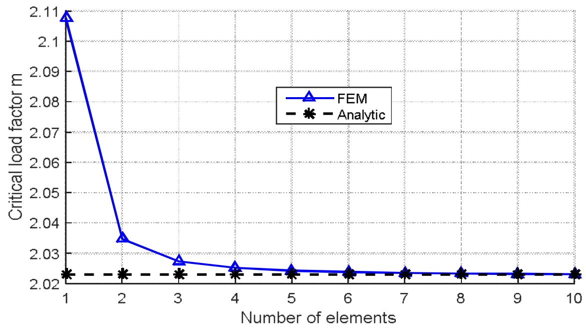

Fig. 4. Comparison of critical load factor between analytical and finite element method.

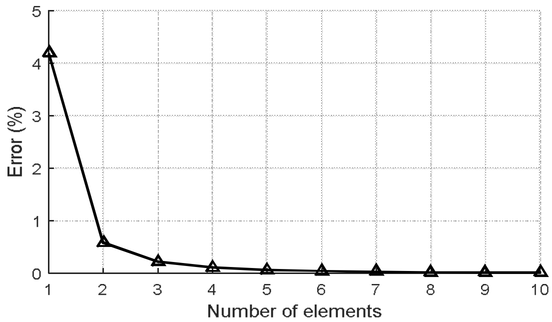

Fig. 5. Error depending on mesh refinement.

As shown in Figure 4, the finite element agreed with the analytical solution if a reasonable number of finite elements was used in the model (approximately more than 10 finite elements). The convergence features of the proposed finite element are shown in Figure 5. The discrepancies between the analytical and finite element solutions of the critical load factor tend rapidly to zero as the mesh is refined.

B. Response Variability of Critical Load Due to the Randomness of Elastic Modulus

A scheme of Monte Carlo Simulation (MCS) was employed to validate the response variability of critical loads of the non-uniform columns. MCS repeats the deterministic analysis on a set of samples of the random field of the elastic modulus. Using the spectral representation proposed in [33], the numerical generation of the homogeneous univariate random field $r(z)$, with zero mean in one dimension, can be generated via the summation formula of cosine functions as follows:

$$\begin{aligned} r(z) &= \sqrt{2}\sum_{i=0}^{N-1} A_i\, cos(\varpi_i z + \varphi_i) \\ A_i &= \sqrt{2S_{rr}(i\Delta\varpi)\Delta\varpi} \\ \Delta\varpi &= \frac{\varpi_u}{N}, \varpi_i = i\Delta\varpi,\ i = 0,1,2,\dots,N-1 \end{aligned} \quad (24)$$

where, ϖ_u, S_{rr} are the upper cut-off frequency and the power spectral density functio, respectively.

The stochastic buckling of linearly tapered cantilever columns with a concrete rectangular cross-section was considered, as shown in Figure 6. The mean modulus of elasticity of the column was assumed to be 33.10^3MPa. The geometric dimensions of the example cross-section columns were: h_0=1m, h_1=0.5 m, H=12m, and b=0.6m.

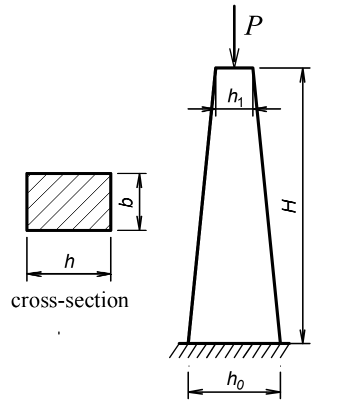

Fig. 6. Linearly tapered cantilever columns with a concrete rectangular cross-section.

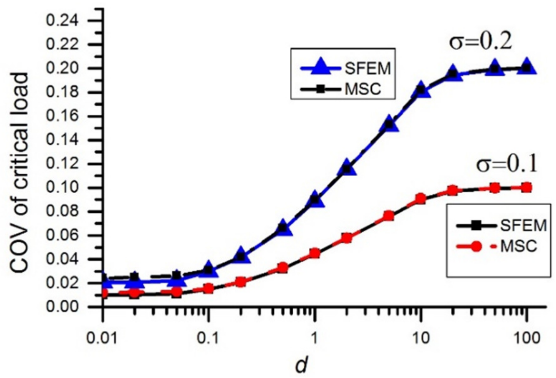

Fig. 7. Effects of the correlation distance d on the different standard deviation σ of the critical load.

Figure 7 shows the comparison of the effect of the correlation distance d of the random field on the variability of the critical load of the proposed formulation with the MCS, for the same cases of the stochastic finite element method. The results of the MCS with 10,000 samples, presented by the dashed-dotted line, denote the corresponding COV of the random field of elastic modulus 0.1 and 0.2. As shown in Figure 6, the response variability of a critical load is converging to COV of the random field of the elastic modulus, as the correlation distance tends to move to infinity in both analysis schemes. As can be observed, the increasing rate of the correlation distances is accelerated with an increase in the coefficient of variation of the random field.

V. CONCLUSION

The paper presented a perturbation technique in conjugation with finite element analysis that was successfully developed for the stochastic buckling problem of non-uniform columns with a random field of elastic modulus. MCS was performed employing 10,000 random samples to simulate the validity of the proposed first-order perturbation solution of the stochastic finite element method. The efficacy of the first-order perturbation method was verified using a homogeneous Gaussian random field by the stochastic finite element method and was in perfect agreement with the MCS, where the correlation distance was as high as 5. The effect of the correlation distance on the response COV of a critical load was obvious, and the response COV increased when correlation distances increased.

ACKNOWLEDGMENT

This research is funded by the University of Transport and Communications (UTC) under grant number T2021-CT-004.

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