# Analysis of Concrete Pavement Slab resting on Non-uniform Elastic Foundation using the Finite Element Method

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**ABSTRACT**

**During the rigid pavement design, the concrete slab working on the base/subbase and subgrade is usually modeled as the slab on the elastic foundation. However, the non-uniform distribution of materials in the base or subgrade layers naturally exists in real conditions and should be considered. In this paper, a finite element method for calculating concrete pavement slabs on an elastic foundation with non-uniform stiffness distribution is developed. This study applies 4-node finite elements and Mindlin plate theory to formulate the finite element equations. The results predicted by the proposed approach are verified analytically. Calculation examples are conducted with the practical settings to investigate the influence of slab and foundation stiffness parameters on the slab displacement.**

***Keywords-pavement slabs; non-uniform foundation; FEM***

## I. INTRODUCTION

There are many studies related to pavements, including rigid pavement and flexible pavement. The design calculations of cement concrete pavement usually use simple models, such as a slab resting on an elastic foundation or a viscoelastic foundation to more complex models, such as a slab resting on a nonlinear foundation or a tensionless foundation. There are many studies on slabs in general, concrete slabs, and concrete pavement slabs on elastic foundations. In fact, the ground is often non-uniform, it has a strong dispersion, so it should be considered during calculations. It can be in the forms of multi-linear [1, 2], nonlinear [3-5], multi-layered [6], and random stiffness [7-9] models.

Authors in [10] analyzed concrete slabs under various subgrade stiffness values using finite element software. Authors in [11] studied experimental full-scale moving loads on asphalt pavement to examine longitudinal strain signals. Authors in [12] studied the buckling of variable thickness nanoplates with considering cracks. Dynamic responses of a functionally graded plate resting on the viscoelastic foundation were analyzed in [13] based on an analytical approach. Authors in [14] computed the dynamic response of sandwich beams on an elastic support. Authors in [15] found the analytical solution for clamped rectangular plates with variable in-plane stiffness. Authors in [16] studied the pavement slabs and considered the temperature and moisture profiles in rigid pavement slabs. Authors in [17] used the phase field to study the cracks on free vibration of rectangular plates with varying thickness. Authors in [18] modeled the slab–foundation interaction on rigid pavement using plate finite elements. Authors in [19] tested concrete pavements subjected to blast loads and compared the results with the ones from finite element commerce software. Authors in [20] developed the ES-MITC3 element for the computation of the natural frequencies of functionally graded porous plates on the partial foundation. Authors in [21] studied the dynamics of runway pavements under wheel loads using ABAQUS. Authors in [22] proposed the arbitrary quadrangular bending finite element for a plate with shear deformations. Authors in [23] studied the effect of linear, parabolic, and sinusoidal Pasternak foundation on bending functionally graduated rectangular plates using analytical solutions. Authors in [24] used the Ritz method to study a rectangular plate resting on non-uniform elastic foundation. Authors in [25] developed the element free Galerkin method for the free vibration of variable thickness plates resting on non-uniform elastic foundations. Authors in [26] studied functionally graded plates under random loads using the isogeometric analysis method.

There are many studies about slabs resting on an elastic foundation. However, new contributions are continually developed and introduced to meet diverse case studies. In this paper, the finite element method for the 4-node finite element is developed to investigate the response of concrete slab on a non-uniform elastic foundation.

## II. FINITE ELEMENT FORMULATION FOR PAVEMENT SLABS ON NON-UNIFORM FOUNDATION

The model of a concrete slab on a non-uniform foundation is shown in Figure 1.

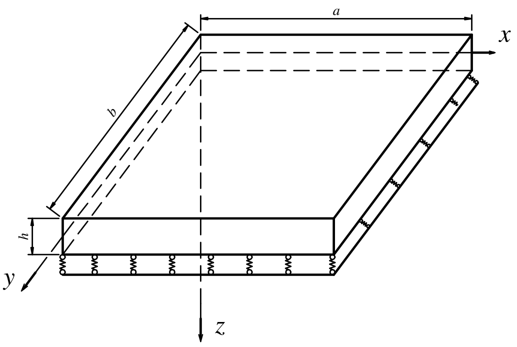


Fig. 1. Model of pavement slab on an elastic foundation.

Displacement of concrete slab is assumed by the Mindlin plate theory [27]:

$$\begin{aligned} u(x,y,z) &= z\theta_x(x,y) \\ v(x,y,z) &= z\theta_y(x,y) \\ w(x,y,z) &= w_o(x,y) \end{aligned} \quad (1)$$

where $\theta x$, $\theta y$ are the rotations of the normal to the middle plane with respect to axes *y* and *x*, respectively.

The strain formulation is:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \end{Bmatrix} \quad (2)$$

Or:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = z \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \end{Bmatrix} \quad (3)$$

$$\{\gamma\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \theta_x + \dfrac{\partial w}{\partial x} \\ \theta_y + \dfrac{\partial w}{\partial y} \end{Bmatrix} \quad (4)$$

The plate finite element model has 4 nodes and 12 degrees of freedom [27]. The stress-strain relations is:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = z \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial x} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \end{Bmatrix} \quad (5)$$

$$\{\gamma\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \theta_x + \dfrac{\partial w}{\partial x} \\ \theta_y + \dfrac{\partial w}{\partial y} \end{Bmatrix} \quad (6)$$

where the stiffness of the material is:

$$[D]_b = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \dfrac{1-\mu}{2} \end{bmatrix} \quad (7)$$

and $[D]_s$ is the transverse shear stresses and strains:

$$[D]_s = \frac{E}{2(1+\mu)} \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \quad (8)$$

The strain energy of the Mindlin slab is:

$$U = \frac{1}{2}\int_V \left( \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}\}[D]_b \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \right) dV + \frac{1}{2}\int_V \left( \{\gamma_{xz} \quad \gamma_{yz}\}[D]_s \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \right) dV \quad (9)$$

The potential energy of the non-uniform foundation is:

$$U_F = \frac{1}{2}\int_V K_w w^2 dV \tag{10}$$

The interpolation functions [27, 28] are:

$$\begin{aligned} N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta) \\ N_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{11}$$

The approximation displacement fields:

$$\begin{aligned} \theta_x^e &= \sum_{i=1}^{4} N_i \theta_i \\ \theta_y^e &= \sum_{i=1}^{4} N_i \theta_i; \\ w_0^e &= \sum_{i=1}^{4} N_i w_i \end{aligned} \tag{12}$$

The displacement vector at the *i*-th node is:

$$\left\{u_i^e\right\} = \begin{Bmatrix} w_0^e \\ \theta_x^e \\ \theta_y^e \end{Bmatrix} \tag{13}$$

The approximate strain by nodal displacements is described by:

$$\mathbf{B}_m = \begin{bmatrix} \frac{\partial N}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N}{\partial y} & 0 \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} & 0 \end{bmatrix} \tag{14}$$

$$\mathbf{B}_s = \begin{bmatrix} 0 & \frac{\partial N}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N}{\partial y} \\ 0 & \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} \end{bmatrix} \tag{15}$$

$$\begin{aligned} \theta_x^e &= \sum_{i=1}^{4} N_i \theta_i \\ \theta_y^e &= \sum_{i=1}^{4} N_i \theta_i; \\ w_0^e &= \sum_{i=1}^{4} N_i w_i \end{aligned} \tag{16}$$

The element stiffness matrices of bending slabs are:

$$\mathbf{B}_f = \begin{bmatrix} \frac{\partial N}{\partial x} & N & 0 \\ \frac{\partial N}{\partial y} & 0 & N \end{bmatrix} \tag{17}$$

The element stiffness matrices of elastic foundation are:

$$\mathbf{B}_f = \begin{bmatrix} \frac{\partial N}{\partial x} & N & 0 \\ \frac{\partial N}{\partial y} & 0 & N \end{bmatrix}; \tag{18}$$

The governing equation is:

$$\boldsymbol{KU} = \boldsymbol{F} \tag{19}$$

where $\boldsymbol{K}$, $\boldsymbol{U}$, $\boldsymbol{F}$ are the stiffness matrix, the displacement matrix, and the load matrix.

## III. NUMERICAL EXAMPLES

At first, in order to verify the reliability of the proposed method, the computation program was developed and performed with numerical input data. The results outputted from the computation would be compared to the results from the analytical solution published in [24]. In this section, a simply supported rectangular plate with dimensions: $a$ = 2 m, $b$ = 2.8 m, and thickness $h$ = 0.1 m. The properties of the plate are: elastic modulus $E = 32\times10^9$ N/m$^2$ and Poisson ratio $\nu$=0.3. The stiffness of the elastic foundation is assumed as linear [24]:

$$\mathrm{K_F} = \mathrm{K_0}\left(1+\gamma\frac{\mathrm{x}}{\mathrm{a}}+\lambda\frac{\mathrm{y}}{\mathrm{b}}\right) \tag{20}$$

The parameter for non-uniform foundation is assumed as $\gamma$ =0.84 and $\lambda$ = 0.6 with the equivalent stiffness of foundation $k_F$ defined as:

$$k_F = \frac{K_0 a^4}{Eh^3} \tag{21}$$

The uniform load on the plate is: $q_0 = 2\times10^4\ \mathrm{N}/\mathrm{m}^2$ .

The calculated results from the proposed method are shown in Table I. Compared to the results computed by analytical method, the differences are very small, less than 1%.

TABLE I. COMPARISON OF THE RESULT WITH THE ANALYTICAL SOLUTION

| $k_F$ | Displacement (mm) | | |
|---|---|---|---|
| | **Present** | **Anal. Sol. [24]** | **Error** |
| 1 | 0.7180 | 0.7117 | 0.877 |
| 2 | 0.6644 | 0.6587 | 0.858 |
| 3 | 0.6181 | 0.6128 | 0.857 |
| 4 | 0.5777 | 0.5728 | 0.848 |
| 5 | 0.5422 | 0.5376 | 0.848 |
| 6 | 0.5106 | 0.5063 | 0.842 |
| 7 | 0.4824 | 0.4783 | 0.850 |
| 8 | 0.4557 | 0.4532 | 0.549 |

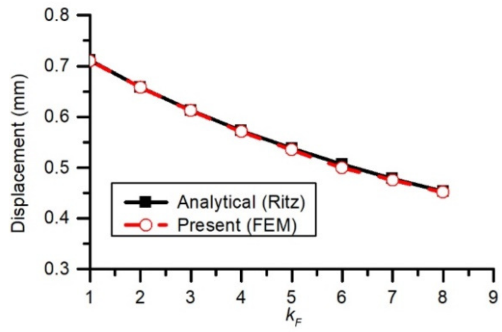


Fig. 2. Effect of foundation on displacement.

Figure 2 shows the displacement at the middle of the slab, with foundation stiffness ranging from 1 to 8. The utilized computation methods are the finite element method and the analytical method. There are only minor differences recorded between the results of the two applied methods. The results of the analytical method are considered to be the theoretical solution. Therefore, the proposed finite element method is relevant and able to derive results with high accuracy.

In the next section, concrete slabs with varying thickness resting on non-uniform and varying stiffness foundation are examined. The concentered application load subjected at the center and middle of long-edge of the slab. The model is shown in Figure 3.

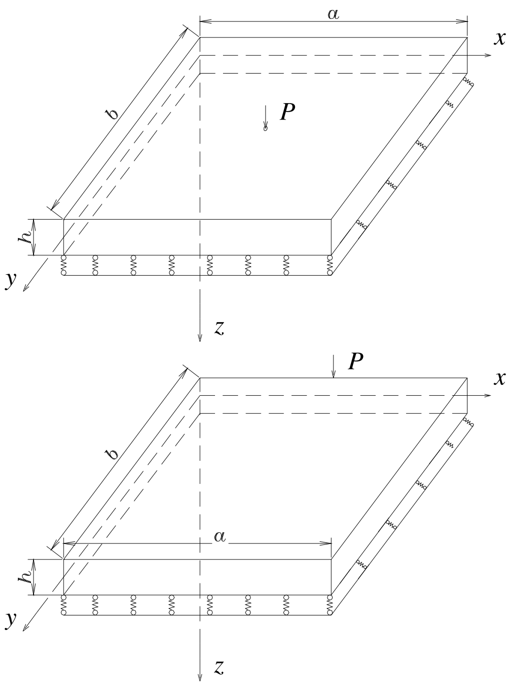


Fig. 3. Concrete pavement slab: Load (a) in the middle of the slab, (b) at the side of the plate.

The dimensions of the slab are $a$ = 4 m, $b$ = 4 m, the plate thickness varies from 20 cm to 40 cm, elastic modulus $E$ = 30Gpa, and Poisson coefficient of 0.25. The stiffness of the foundation will be investigated in the range of 50 to 150 MPa. The concentrated forces at central or edge of slabs of slab $P$ = 100 kN. The parameter for non-uniform foundation is assumed as $\gamma$ =0.5 and $\lambda$ = 0.5.

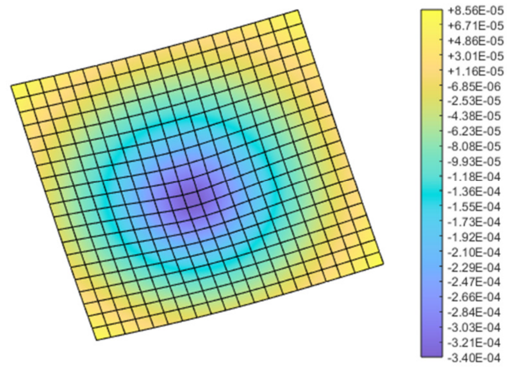

Fig. 4. Displacements by concentrated forces at central slabs with h = 0.2 m, $K_0$ = 50MN/m$^2$.

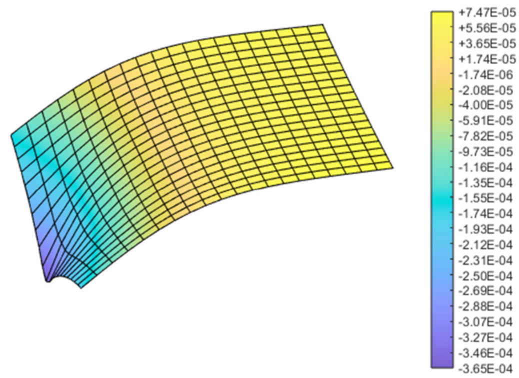

Fig. 5. Displacements by concentrated forces at central slabs with h = 0.4 m, $K_0$ = 50MN/m$^2$.

Table II shows the displacement at the center of the pavement slab when the force is concentrated there. The data from Table II show that the deflection decreases markedly with increasing thickness of the slab and increasing stiffness of the foundation. When the load is applied at the symmetric center of the slabs, the effect of non-uniform stiffness of the foundation on the displacement is quite significant. The displacements in the stiffer portion of the foundation significantly decrease.

TABLE II. DISPLACEMENTS BY CONCENTRATED FORCES AT THE CENTER OF THE SLABS

| *Kw* (MPa) | Displacements ($10^{-4}$ m) at the thickness of the plate | | |
|---|---|---|---|
| | 20 (cm) | 30 (cm) | 40 (cm) |
| 50 | 3.397 | 2.040 | 1.496 |
| 100 | 2.320 | 1.393 | 0.981 |
| 150 | 1.857 | 1.122 | 0.785 |
| 200 | 1.588 | 0.963 | 0.674 |

Table III shows the maximum displacements of the plate when subjected to a concentrated load placed at the edge. This displacement corresponds to the location of the applied concentrated load. Table III shows a significant reduction in the displacement as the thickness of the slabs and the stiffness of the foundation increase. The maximum displacements of the slabs in this case are much larger compared to when the load is applied at the center of the slabs.

TABLE III. MAXIMUM DISPLACEMENTS BY CONCENTRATED FORCES AT THE EDGE OF THE SLABS

| ***Kw*** **(MPa)** | **Displacements ($10^{-4}$ m) at thickness of plate** | | |
|---|---|---|---|
| | **20 (cm)** | **30 (cm)** | **40 (cm)** |
| 50 | 13.232 | 6.075 | 3.647 |
| 100 | 10.634 | 4.829 | 2.797 |
| 150 | 9.384 | 4.249 | 2.440 |
| 200 | 8.591 | 3.886 | 2.224 |

## IV. CONCLUSIONS

The finite element method, using 4-node elements for slabs on non-uniform foundation, was successfully developed in this paper. The governing equation of the finite elements is formulated with the help of the Mindlin plate theory. The numerical examples show that the results derived from the proposed method have very small differences with the results of the analytical method, proving the high accuracy of the proposed finite element procedure. For a simply supported or free edge slab, when the stiffness of the foundation is increased, the displacement clearly decreases. The numerical calculation results also demonstrate a significant influence of the non-uniform stiffness of the foundation on the displacement of the slabs.

## REFERENCES


[1] D. Froio, E. Rizzi, F. M. F. Simoes, and A. Pinto Da Costa, "Dynamics of a beam on a bilinear elastic foundation under harmonic moving load," *Acta Mechanica*, vol. 229, no. 10, pp. 4141–4165, Oct. 2018, https://doi.org/10.1007/s00707-018-2213-4.

[2] S. Lenci, "Propagation of periodic waves in beams on a bilinear foundation," *International Journal of Mechanical Sciences*, vol. 207, Oct. 2021, Art. no. 106656, https://doi.org/10.1016/j.ijmecsci.2021.106656.

[3] F. W. Beaufait and P. W. Hoadley, "Analysis of elastic beams on nonlinear foundations," *Computers & Structures*, vol. 12, no. 5, pp. 669–676, Nov. 1980, https://doi.org/10.1016/0045-7949(80)90168-6.

[4] G. C. Tsiatas, "Nonlinear analysis of non-uniform beams on nonlinear elastic foundation," *Acta Mechanica*, vol. 209, no. 1, pp. 141–152, Jan. 2010, https://doi.org/10.1007/s00707-009-0174-3.

[5] D. Xuan Quy and N. Thị, "Static analysis of beam resting on elastic foundation by anisotropic beam-foundation element taking into account non-contact between beam and foundation," *Transport and Communications Science Journal*, vol. 72, no. 5, pp. 552–564, Jun. 2021, https://doi.org/10.47869/tcsj.72.5.4.

[6] O. Khalaj, S. N. M. Tafreshi, B. Masek, and A. Dawson, "Improvement of pavement foundation response with multi-layers of geocell reinforcement: cyclic plate load test," *Geomechanics and Engineering*, vol. 9, no. 3, pp. 373–395, Sep. 2015, https://doi.org/10.12989/gae.2015.9.3.373.

[7] T. D. Hien, N. D. Hung, N. T. Kien, and H. C. Noh, "The variability of dynamic responses of beams resting on elastic foundation subjected to vehicle with random system parameters," *Applied Mathematical Modelling*, vol. 67, pp. 676–687, Mar. 2019, https://doi.org/10.1016/j.apm.2018.11.018.

[8] S. Mohammadzadeh, M. Esmaeili, and M. Mehrali, "Dynamic response of double beam rested on stochastic foundation under harmonic moving load," *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 38, no. 6, pp. 572–592, 2014, https://doi.org/10.1002/nag.2227.

[9] S. Mohammadzadeh and M. Mehrali, "Dynamic response of ladder track rested on stochastic foundation under oscillating moving load," *Journal of Theoretical and Applied Mechanics*, vol. 55, no. 1, pp. 281–291, 2017, https://doi.org/10.15632/jtam-pl.55.1.281.

[10] A. S. Brand and J. R. Roesler, "Finite element analysis of a concrete slab under various non-uniform support conditions," *International Journal of Pavement Engineering*, vol. 15, no. 5, pp. 460–470, May 2014, https://doi.org/10.1080/10298436.2013.837463.

[11] L. X. Quy, N. M. Lan, N. Q. Tuan, and H. Pierre, "Longitudinal strain analysis in asphalt pavement under full-scale moving loads," *Transport and Communications Science Journal*, vol. 72, no. 4, pp. 359–370, 2022.

[12] D. H. Duc, D. V. Thom, and P. M. Phuc, "Buckling analysis of variable thickness cracked nanoplates considerting the flexoelectric effect," *Transport and Communications Science Journal*, vol. 73, no. 5, pp. 470–485, 2022.

[13] D. T. Thuy, L. N. Ngoc, D. N. Tien, and H. V. Thanh, "An Analytical Solution for the Dynamics of a Functionally Graded Plate resting on Viscoelastic Foundation," *Engineering, Technology & Applied Science Research*, vol. 13, no. 1, pp. 9926–9931, Feb. 2023, https://doi.org/10.48084/etasr.5420.

[14] T. D. Hien, N. D. Hung, N. T. Hiep, G. V. Tan, and N. V. Thuan, "Finite Element Analysis of a Continuous Sandwich Beam resting on Elastic Support and Subjected to Two Degree of Freedom Sprung Vehicles," *Engineering, Technology & Applied Science Research*, vol. 13, no. 2, pp. 10310–10315, Apr. 2023, https://doi.org/10.48084/etasr.5464.

[15] T. C. Yu, G. J. Nie, Z. Zhong, F. Y. Chu, and X. J. Cao, "Analytical Solution of the Bending Problem for Rectangular Orthotropic Plates with a Variable in-Plane Stiffness," *Mechanics of Composite Materials*, vol. 57, no. 1, pp. 115–124, Mar. 2021, https://doi.org/10.1007/s11029-021-09938-1.

[16] K. Huang, D. G. Zollinger, X. Shi, and P. Sun, "A developed method of analyzing temperature and moisture profiles in rigid pavement slabs," *Construction and Building Materials*, vol. 151, pp. 782–788, Oct. 2017, https://doi.org/10.1016/j.conbuildmat.2017.06.120.

[17] P. M. Phuc, "Using phase field and third-order shear deformation theory to study the effect of cracks on free vibration of rectangular plates with varying thickness," *Transport and Communications Science Journal*, vol. 71, no. 7, pp. 853–867, 2020.

[18] A. Zokaei-Ashtiani, C. Carrasco, and S. Nazarian, "Finite element modeling of slab–foundation interaction on rigid pavement applications," *Computers and Geotechnics*, vol. 62, pp. 118–127, Oct. 2014, https://doi.org/10.1016/j.compgeo.2014.07.003.

[19] B. M. Luccioni and M. Luege, "Concrete pavement slab under blast loads," *International Journal of Impact Engineering*, vol. 32, no. 8, pp. 1248–1266, Aug. 2006, https://doi.org/10.1016/j.ijimpeng.2004.09.005.

[20] P.-C. Nguyen, Q. H. Pham, T. T. Tran, and T. Nguyen-Thoi, "Effects of partially supported elastic foundation on free vibration of FGP plates using ES-MITC3 elements," *Ain Shams Engineering Journal*, vol. 13, no. 3, May 2022, Art. no. 101615, https://doi.org/10.1016/j.asej.2021.10.010.

[21] C. Liu *et al.*, "Numerical Analysis on the Mechanical Properties of the Concrete Precast Pavement of Runways under the Wheel Load," *Applied Sciences*, vol. 12, no. 19, Jan. 2022, Art. no. 9826, https://doi.org/10.3390/app12199826.

[22] Y. Tyukalov, "Arbitrary quadrangular finite element for plates with shear deformations," *Magazine of Civil Engineering*, vol. 107, no. 7, 2021, Art. no. 10707, https://doi.org/10.34910/MCE.107.7.

[23] B. Bouderba and B. Hamza Madjid, "Bending analysis of P-FGM plates resting on nonuniform elastic foundations and subjected to thermo-mechanical loading," *Cogent Engineering*, vol. 9, no. 1, Dec. 2022, Art. no. 2108576, https://doi.org/10.1080/23311916.2022.2108576.

[24] T. D. Hien and B. T. Quang, "Analysis of isotropic rectangular plate resting on non-uniform elastic foundation using Ritz approach," *Materials Today: Proceedings*, vol. 19, pp. 158–160, Jan. 2019, https://doi.org/10.1016/j.matpr.2019.06.631.

[25] E. Bahmyari and A. Rahbar-Ranji, "Free vibration analysis of orthotropic plates with variable thickness resting on non-uniform elastic foundation by element free Galerkin method," *Journal of Mechanical Science and Technology*, vol. 26, no. 9, pp. 2685–2694, Sep. 2012, https://doi.org/10.1007/s12206-012-0713-z.

[26] N. T. Hiep, D. S. Dan, N. D. Diem, and D. N. Tien, "NURBS-based Isogeometric Analysis and Refined Plate Theory Application on a Functionally Graded Plate Subjected to Random Loads," *Engineering, Technology & Applied Science Research*, vol. 13, no. 2, pp. 10243–10248, Apr. 2023, https://doi.org/10.48084/etasr.5478.

[27] A. J. M. Ferreira, *MATLAB Codes for Finite Element Analysis: Solids and Structures*. Dordrecht, Netherlands: Springer, 2009.

[28] J. Reddy, *An Introduction to the Finite Element Method*, 3rd edition. New York, NY, USA: McGraw-Hill Education, 2005.