Finite Element Analysis for the Free Vibration of a Rigid Pavement resting on a Non-uniform Elastic Foundation

Bui Tuan Anh
University of Transport and Communications, Vietnam
tuananh.bui@utc.edu.vn

Nguyen Trong Hiep
University of Transport and Communications, Vietnam
nguyentronghiep@utc.edu.vn

Le Vinh An
University of Transport and Communications, Vietnam
levinhan@utc.edu.vn

Nguyen Van Lap
University of Transport and Communications, Vietnam
lapnguyendsb@utc.edu.vn (corresponding author)

Received: 13 May 2023 | Revised: 8 June 2023 | Accepted: 29 June 2023

 DOI: https://doi.org/10.48084/etasr.6039

ABSTRACT

This paper presents a finite element analysis of the free vibration behavior of rigid pavements resting on non-uniform foundations. The rigid pavement was modeled using the Mindlin plate theory, while the supporting soil medium was approximated by a Winkler model with non-uniform stiffness. A finite element formulation was established to govern the equation of free vibration for rigid pavements. Subsequently, a computer program was developed based on the proposed algorithm, enabling the determination of natural frequencies and mode shapes. The accuracy of the proposed method was verified by comparing numerical examples of free vibration with analytical results. These numerical examples also demonstrate the significant influence of the foundation stiffness on natural frequencies and mode shapes.

Keywords-rigid pavement; FEM; free vibration; nonuniform foundation

I. INTRODUCTION

Rigid pavements offer numerous advantages, such as durability, longevity, good resistance to adverse environmental conditions [1], heavy load-bearing capacity, and high cycle fatigue durability. These qualities make them a suitable choice for airports and roads with heavy loads or high traffic density. Despite their outstanding characteristics, rigid pavements also have some drawbacks during operation. Except for continuously reinforced concrete pavements, rigid pavements contain joints to accommodate the expansion and contraction of concrete caused by temperature variations. The purpose of these joints is to prevent the development of curling and warping stresses in the slab, which can cause cracks or other forms of damage. Additionally, joints are often included due to limitations in construction technology. The arrangement of the transverse joints can cause uneven vehicle movement and generate noise. Furthermore, when joint sealing compounds age, stormwater can infiltrate the foundation or subgrade, potentially altering the physical-mechanical properties of the base materials and/or subgrade soil. Consequently, this can cause a non-uniform load-bearing capacity of the foundation. Focussing on the noise defect of rigid pavements it can be assumed that their high stiffness and the presence of transverse joints result in greater noise emissions compared to other pavement types. The process of noise emission on the pavement is complicated and derives from numerous sources. However, it is partially attributed to the vibration of the pavement slabs and the interaction between the tire and the pavement surface. The analysis of foundation-supported structures has long been a popular topic in construction. Various types of structures have attracted interest, including beams on elastic foundations [2-4], beams on viscoelastic foundations, plates on elastic foundations [5-6], and plates on viscoelastic foundations [7-9]. Pavement slabs can be modeled as either thin or thick structures. Depending on the type of problem, the analysis of plate structures can employ various methods including analytical approaches [10-11], semi-analytic techniques, numerical methods such as finite element analysis [3, 12-15], and isogeometric analysis [16-17]. In [18], the dynamic responses of pavement slabs resting on foundations were studied using a state space approach. In [19], the eigenproblems of plates with variable thicknesses supported by non-uniform elastic foundations were investigated using the element-free Galerkin method. In [20], Fourier and Laplace transforms were employed to solve the dynamic problems of concrete pavements subjected to impact loading. In [21], a trigonometric series approximation was used to determine the displacement of rectangular thick plates on the Pasternak foundation with free boundary edges. In [22], the crack behavior of cement concrete pavements was analyzed using the finite element software ILLI-SLAB. In [23], three-dimensional finite element analysis was used to study the dynamics of rigid pavements under moving loads.

Although several studies explored rigid pavements supported by elastic non-foundation, this study aims to develop a practical model applicable to engineers' design practices. Finite element formulations were developed for analyzing rigid pavements resting on non-uniform elastic foundations. The current analysis employed four-node quadrilateral elements and utilized displacement field assumptions based on the Mindlin plate theory. MATLAB was used to implement and execute the algorithms.

II. FINITE ELEMENT MODEL FOR RIGID PAVEMENT ON THE NONUNIFORM FOUNDATION

The model was a rigid pavement slab placed on a foundation, which includes layers of base/subbase and subgrade and exhibits non-uniform stiffness. This configuration is represented as a plate resting on a Winkler foundation, as illustrated in Figure 1.

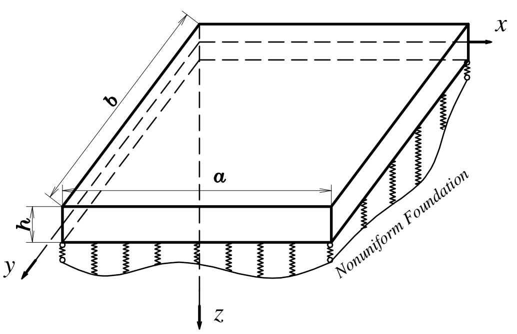

Fig. 1. Model of pavement slab on elastic foundation.

The computation incorporates the displacement field based on the first-order plate theory, commonly referred to as the Mindlin plate theory [24]:

$$u(x,y,z) = z\theta_x(x,y)$$
$$v(x,y,z) = z\theta_y(x,y) \quad (1)$$
$$w(x,y,z) = w_o(x,y)$$

where w_0 represents the displacement along the z-axis, while θ_x and θ_y denote the rotations of the normal to the middle plane around the y and x axes, respectively.

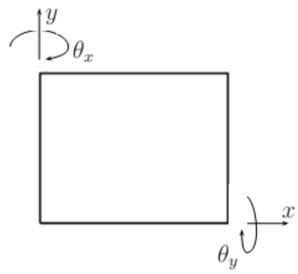

Fig. 2. The rotations θ_x and θ_y in the Mindlin plate theory.

The strain formulation can be expressed as follows:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = z \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{Bmatrix} \quad (3)$$

$$\{\gamma\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix} \quad (4)$$

The stress-strain relations can be defined as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (5)$$

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{2(1+\mu)} \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (6)$$

where the stiffness of the material constituents is:

$$[D]_b = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \quad (7)$$

$$[D]_s = \frac{E}{2(1+\mu)} \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} \quad (8)$$

The strain energy of the Mindlin slab is given as:

$$U = \frac{1}{2}\int_V \left(\{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy}\}[D]_b \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \right) dV + \frac{1}{2}\int_V \left(\{\gamma_{xz} \quad \gamma_{yz}\}[D]_s \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \right) dV \quad (9)$$

The potential energy of the non-uniform foundation is expressed as:

$$U_F = \frac{1}{2}\int_V K_w w^2 dV \quad (10)$$

The interpolation functions [24-25] are used for the quadrilateral element:

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta) \quad (11)$$
$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

The displacement fields are approximated as:

$$\theta_x^e = N_1\theta_{x1}^e + N_2\theta_{x2}^e + N_3\theta_{x3}^e + N_4\theta_{x4}^e$$
$$\theta_y^e = N_1\theta_{y1}^e + N_2\theta_{y2}^e + N_3\theta_{y3}^e + N_4\theta_{y4}^e \quad (12)$$
$$w_0^e = N_1w_1^e + N_2w_2^e + N_3w_3^e + N_4w_4^e$$

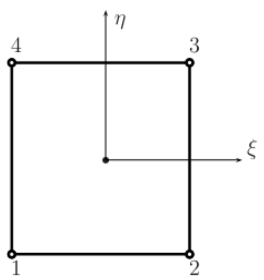

Fig. 3. Quadrilateral element in natural coordinates.

The displacement vector at the *i*-th node is represented as:

$$\{u_i^e\} = \begin{Bmatrix} w_0^e \\ \theta_x^e \\ \theta_y^e \end{Bmatrix} \quad (13)$$

Approximate strain is obtained using nodal displacements:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N}{\partial y} & 0 \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} w_0 \\ \theta_x \\ \theta_y \end{Bmatrix} \quad (14)$$

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} 0 & \frac{\partial N}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N}{\partial y} \\ 0 & \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} \end{bmatrix} \begin{Bmatrix} w_0 \\ \theta_x \\ \theta_y \end{Bmatrix} \quad (15)$$

$$w = [N \quad 0 \quad 0] \begin{Bmatrix} w_0 \\ \theta_x \\ \theta_y \end{Bmatrix} \quad (16)$$

The bending strain stiffness matrix is defined as:

$$B_m = \begin{bmatrix} \frac{\partial N}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N}{\partial y} & 0 \\ \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} & 0 \end{bmatrix} \quad (17)$$

The shear strain stiffness matrix is given by:

$$B_s = \begin{bmatrix} 0 & \frac{\partial N}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N}{\partial y} \\ 0 & \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} \end{bmatrix} \quad (18)$$

Element stiffness matrices for bending in rigid pavements are:

$$K_m = \int_\Omega B_m^T[D]_b B_m d\Omega \quad (19)$$

The element stiffness matrices for shear in rigid pavements are:

$$K_s = \int_\Omega B_s^T[D]_s B_s d\Omega \quad (20)$$

The element stiffness matrices for the elastic foundation are:

$$K_f = \int_\Omega B_f^T K_w B_f d\Omega \quad (21)$$

The stiffness matrices of the elements are:

$$K^e = K_m + K_s + K_f \quad (22)$$

The element mass matrices are:

$$M^e = \int_\Omega \begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix} \begin{bmatrix} m_0 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix} d\Omega \quad (23)$$

The forced vibration equation is given by:

$$[M]\{\ddot{U}\} + [K]\{U\} = \{F\} \quad (24)$$

where *K*, *U*, and *F* represent the stiffness, displacement, and load matrices, respectively. The free vibration equation is expressed as:

$$[M]\{\ddot{U}\} + [K]\{U\} = \{0\} \quad (25)$$

The displacement is represented in terms of a harmonic function, given as:

$$\{U\} = \begin{Bmatrix} u_1 \\ u_2 \\ \\ u_{ne} \end{Bmatrix} sin(\omega t + \varphi) \quad (26)$$

By substituting (26) into (25), an eigenvalue problem can be derived to determine the natural frequencies:

$$([K] - \omega^2[M])\{U\} = \{0\} \quad (27)$$

III. NUMERICAL EXAMPLES

A. Example 1: Validation examples

The natural frequencies of a simply supported rectangular plate on an elastic foundation obtained with the proposed method were compared with the exact analytical results derived using an analytical method [27]. A simply supported rectangular plate was considered, with a = 2 m, b = 2 m, thickness h = 0.12 m, and elastic modulus E = 30 GPa, mass density ρ = 2400kg/m^3, and Poisson's ratio ν = 0.25. The plate was discretized into a grid of 10×10 finite elements. The equivalent stiffness of the foundation is defined as:

$$k_F = \frac{K_0 a^4}{Eh^3} \quad (28)$$

Table I presents the computed results of the rigid pavement natural frequencies using the proposed method. A comparison with the exact solutions obtained by an analytical method [27] revealed very small differences, all less than 0.4%. Figure 4 visualizes modes 1 and 2.

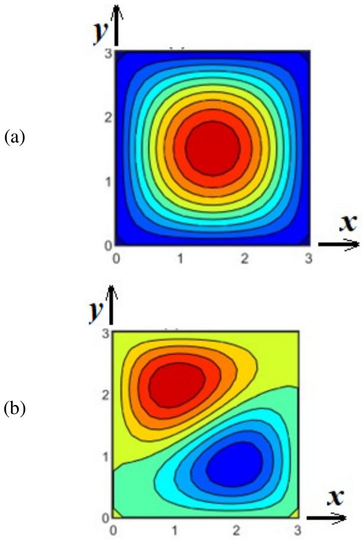

Fig. 4. (a) First and (b) second mode shapes.

TABLE I. COMPARISON OF PROPOSED METHODS' RESULTS WITH EXTRACT SOLUTION

k_F	Mode	Natural frequencies (rad/s)		
		Present	Analytical method [27]	Error (%)
1	Mode 1	280.6415	281.4026	0.2705
	Mode 2	692.5564	695.1653	0.3753
5	Mode 1	296.0160	296.7765	0.2563
	Mode 2	698.9040	701.5295	0.3743
10	Mode 1	314.1780	314.9404	0.2421
	Mode 2	706.7583	709.4045	0.3730
20	Mode 1	347.6671	348.4389	0.2215
	Mode 2	722.2106	724.8979	0.3707

B. Example 2

To investigate the impact of a non-uniform foundation on the free vibration of a rigid pavement, a slab with dimensions a = 3 m, b = 4 m, and a thickness of 16 cm was considered. This slab was characterized by an elastic modulus E = 30 Gpa and a Poisson's coefficient of 0.25. The boundary condition of the rigid pavement plate consists of four free edges. The stiffness of the non-uniform elastic foundation is assumed as follows:

Foundation 1: linear variation:

$$K_F = K_0\left(1+\gamma\frac{x}{a}+\lambda\frac{y}{b}\right) \tag{29}$$

Foundation 2: nonlinear variation:

$$K_F = K_0\left\{1+\gamma\left(0.5-\frac{x}{a}\right)^2+\lambda\left(0.5-\frac{y}{b}\right)^2\right\} \tag{30}$$

The parameters for the non-uniform foundation were assumed as $k_F = 10$, $\gamma = 0.4$, $\lambda = 0.4$. Figure 5 illustrates the first three mode shapes of a simply supported pavement plate on a non-uniform linear elastic foundation, with stiffness parameter $k_F = 10$. Figure 5(b),(c) shows the loss of asymmetry in the mode shapes due to the nonuniform stiffness of the foundation.

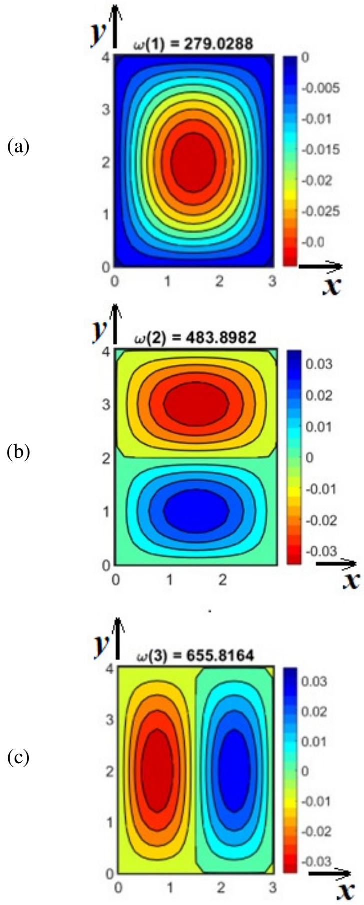

Fig. 5. Modes (a) 1, (b) 2, and (c) 3 with linear elastic foundation k_F = 10.

Table II depicts the calculated results of the first three natural frequencies for a simply supported pavement plate on a foundation, considering two types of non-uniform foundations: linear and nonlinear. Numerical analysis reveals that the frequencies associated with a linearly varying foundation were higher compared to those associated with a quadratic variation.

TABLE II. NATURAL FREQUENCIES OF RIGID PAVEMENT ON NONUNIFORM FOUNDATION

k_F		Natural frequencies (rad/s)		
		Mode 1	Mode 2	Mode 3
1	Foundation 1	223.47	454.17	634.24
	Foundation 2	221.61	453.29	633.62
10	Foundation 1	279.03	483.90	655.83
	Foundation 2	263.81	475.62	649.75
20	Foundation 1	329.90	514.93	678.98
	Foundation 2	303.90	499.26	667.22

IV. CONCLUSIONS

This paper presented a finite element formulation for analyzing the free vibration behavior of a rigid pavement resting on a non-uniform elastic foundation. The stiffness of the non-uniform elastic foundation was assumed to exhibit linear or quadratic variation within the rigid pavement plane. MATLAB was employed to implement and execute the algorithms developed for computing the natural frequencies and mode shapes of the free vibrations. The numerical results clearly illustrated that the mode shape of the vibration undergoes significant changes when the foundation stiffness varies.

ACKNOWLEDGMENT

This research is funded by University of Transport and Communications (UTC) under grant number T2024-CT-027.

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