Stability of a Non-uniform Column resting on a Foundation, calculated with the Finite Element Method

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ABSTRACT

This paper presents the application of the finite element method to columns with non-uniform cross-sections resting on elastic foundations, solving the eigenvalue problem of finding the critical load. The formula for calculating the stiffness matrix resulting from column bending, elastic foundation, and geometric stiffness is established based on the principle of virtual work. Based on the finite element formulas, an algorithm is established in MATLAB to find the column's critical force. The results obtained using the proposed approach agree with the exact solution obtained with analytical methods. In many cases, the calculation results of the critical force are given to assess the effects of the foundation's stiffness and boundary condition on the critical load.

Keywords-column; FEM; non-uniformity; stability; foundation

I. INTRODUCTION

Nowadays, there are many available high-strength and lightweight materials used to create many types of slender structures for mechanical and construction applications. Nevertheless, slender structures are prone to instability. For instance, bridges have many components subjected to compression, such as trusses and towers. These structures are commonly designed with variable cross-sections to suit the load-bearing characteristics of the components and avoid material wastage. The stability of structures with variable cross-sections has been studied, among others, in [1-3]. Some researchers have evaluated the stability of bars with variable cross-sections, using analytical methods to find exact or approximate solutions. Author in [4] found the exact solution for certain columns with variable cross-sections and elastic connections, subjected to distributed axial loads. Authors in [5] found the exact solution for the buckling problem of a non-homogeneous Euler-Bernoulli column, using hypergeometric and elementary functions. Author in [6] used an approximate form solution applying a polynomial series to determine the stability of columns with variable cross-sections, subjected to axial loads. In [7], the stability of columns with variable cross-sections and rigid and elastic connections was calculated by approximately solving the differential equation of stability. Authors in [8] evaluated columns with varying stepped cross-sections, using the Rayleigh-Ritz method and authors in [9] studied the stability of reinforced concrete columns with elastic connections.

Analytical methods have limitations in assessing complex structures, thus, many authors attempted to use numerical approaches like the finite element method to determine structural stability. Authors in [10] used a numerical method to investigate the nonlinear stability of axially functionally graded columns. Authors in [11] performed the bending-shear buckling of a sandwich beam, using approximate analytical solutions. Authors in [12] used a numerical method to investigate composite rods with semi-rigid nonlinear deformable connections. Authors in [13] used the finite element method to study the elastic-plastic stability of bars. Authors in [14] used the Galerkin finite element method to assess the stability of bars with variable cross-sections. Author in [15] used the finite element method to estimate the stability of columns with variable cross-sections (e.g. steps and cracks). However, these studies have limitations due to the diversity of bars with variable cross-sections. The finite element method shows versatility in evaluating structures. However, there is only a limited number of studies on the stability of bars with variable cross-sections using the finite element method. Authors in [16] developed the ES-MITC3 triangular element to buckle functionally graded porous plates with variable thickness. Moreover, authors in [17, 18] studied the random stability of columns with variable cross-sections, using the random finite element method.

This study proposes a finite element method for a column with a cross-section supported on an elastic foundation. The principle of virtual work was applied to establish the system of governing equations. MATLAB was used to calculate the critical force value.

II. FINITE ELEMENT FORMULATION FOR A COLUMN RESTING ON AN ELASTIC FOUNDATION AND HAVING A NON-UNIFORM CROSS-SECTION

Let us consider a column with any variable cross-section bearing a concentrated force at the top of the column, with elastic connections at both ends (Figure 1).

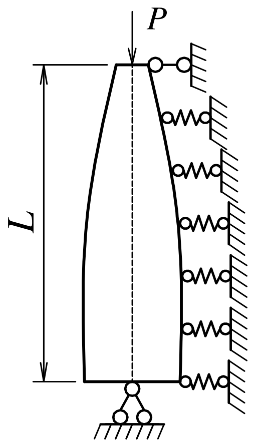


Fig. 1. Non-uniform column resting on an elastic foundation.

Although the column has a variable cross-section, the cross-section is assumed to be symmetrical, and the column centroid is a straight line. The flexural stiffness *EI* of the element holding the two nodes is linearly approximated, using the finite element model.

The displacement of the element along the z-axis is approximated by Hermite functions [19, 20]:

$$w_e = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} N \end{bmatrix} \{q\} \quad (1)$$

The potential energy of deformation resulting from column bending is expressed as:

$$\Pi = \frac{1}{2}\int_0^L EI(z)\left(\frac{d^2w}{dz^2}\right)^2 dz \quad (2)$$

The potential energy of the elastic foundation is calculated by:

$$\begin{aligned} \Pi_{dh} &= \frac{1}{2}\int_0^L K_{dh} w^2 dz \\ &= \frac{1}{2}\sum_e \int_0^{L_e} K_{dh} \left(\begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \right)^2 dz \end{aligned} \quad (3)$$

Substituting the displacement approximation (1) into the potential energy functions (3), (4), and (5), and then using the variational principle, we obtain the element stiffness matrix and the geometric stiffness matrix, which are combined to obtain the structure stiffness matrix.

The stiffness matrix of the element due to bending can be expressed by (4):

$$[K_{be}] = \begin{bmatrix} \frac{6(EI_{1e}+EI_{2e})}{L_e^3} & \frac{4EI_{1e}+2EI_{2e}}{L_e^2} & -\frac{6(EI_{1e}+EI_{2e})}{L_e^3} & \frac{2EI_{1e}+4EI_{2e}}{L_e^2} \\ & \frac{3EI_{1e}+EI_{2e}}{L_e} & -\frac{4EI_{1e}+2EI_{2e}}{L_e^2} & \frac{EI_{1e}+EI_{2e}}{L_e} \\ & & \frac{6(EI_{1e}+EI_{2e})}{L_e^3} & -\frac{2EI_{1e}+4EI_{2e}}{L_e^2} \\ \text{Sym} & & & \frac{EI_{1e}+3EI_{2e}}{L_e} \end{bmatrix} \quad (4)$$

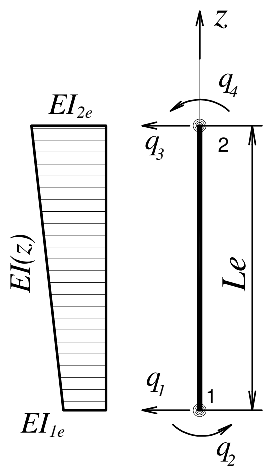


Fig. 2. Finite element model approximation for a non-uniform column.

The geometric stiffness matrix is expressed by (5) and the element stiffness matrix of the elastic foundation is expressed by (6):

$$\left[K_{ge}\right] = P\begin{bmatrix} \frac{6}{5L_e} & \frac{1}{10} & -\frac{6}{5L_e} & \frac{1}{10} \\ & \frac{2}{15L_e} & -\frac{1}{10} & -\frac{1}{30L_e} \\ & & \frac{6}{5L_e} & -\frac{1}{10} \\ \text{Sym} & & & \frac{2}{15L_e} \end{bmatrix} \quad (5)$$

$$\left[K_f\right] = K_{dh}\begin{bmatrix} \frac{13}{35}L_e & \frac{11}{210}L_e^2 & \frac{9}{70}L_e & -\frac{13}{420}L_e^2 \\ & \frac{1}{105}L_e^3 & \frac{13}{420}l^2 & -\frac{1}{140}L_e^3 \\ & & \frac{13}{35}L_e & -\frac{11}{210}L_e^2 \\ sym & & & \frac{1}{105}L_e^3 \end{bmatrix} \quad (6)$$

The column stability equation is:

$$\left[K_b + K_f - \Lambda K_g\right]\{U\} = \{0\} \quad (7)$$

where K_b, K_f, K_g, U are the bending stiffness matrix, the foundation's stiffness matrix, the geometric stiffness matrix, and the displacement vector, respectively.

The critical force is determined from the minimum value of the eigenvalue Λ.

III. NUMERICAL EXAMPLES

A. *Example 1: Validation Example*

To validate the current approach, consider a free-clamped column with a variable cross-section, as shown in Figure 3.

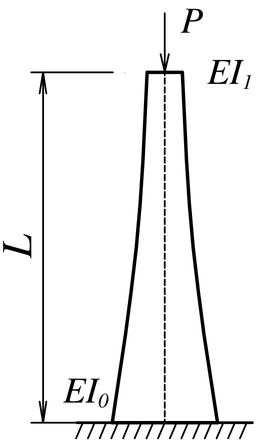


Fig. 3. Non-uniform column.

In this example, the stiffness ratio between both ends of the column is calculated as:

$$\frac{EI_1}{EI_0} = \alpha \quad (8)$$

The bending stiffness of the column follows the below expression:

$$EI = \frac{EI_0}{\alpha}\left(\frac{a + L - z}{a}\right)^2 \quad (9)$$

where:

$$a = \frac{L}{\frac{1}{\sqrt{\alpha}} - 1} \quad (10)$$

For a convenient comparison and checking of results, we calculate the stability coefficient k of the column through a dimensionless quantity, as follows:

$$k = \frac{P_{cr}L^2}{EI_0} \quad (11)$$

The critical force is usually expressed as a normalized formula with the geometric length factor λ:

$$P_{cr} = \frac{\pi^2 EI_0}{(\lambda L)^2} \quad (12)$$

Thus, the geometric length factor is calculated as follows:

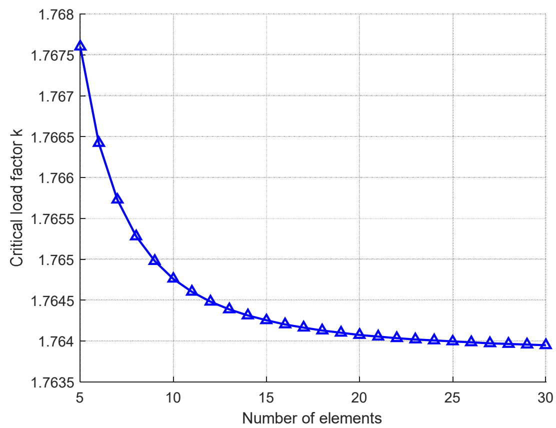
$$\lambda = \frac{\pi}{L}\sqrt{\frac{EI_0}{P_{cr}}} \quad (13)$$

The error between the exact result and the finite element solution is estimated by:

$$Err = 100 \times \frac{k_{FEM} - k_{Exact}}{k_{Exact}}(\%) \quad (14)$$

Figures 4 and 5 illustrate the value of the stability coefficient k and error versus the number of elements.

(a)



(b)

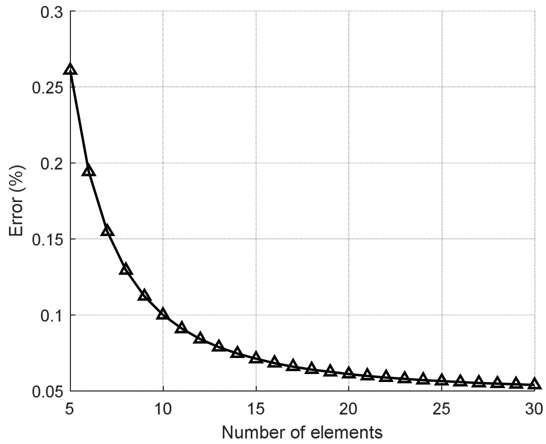


Fig. 4. Convergence of the critical force and number of elements when the stiffness ratio is α = 0.3. (a) Stability coefficient *k*, (b) error.

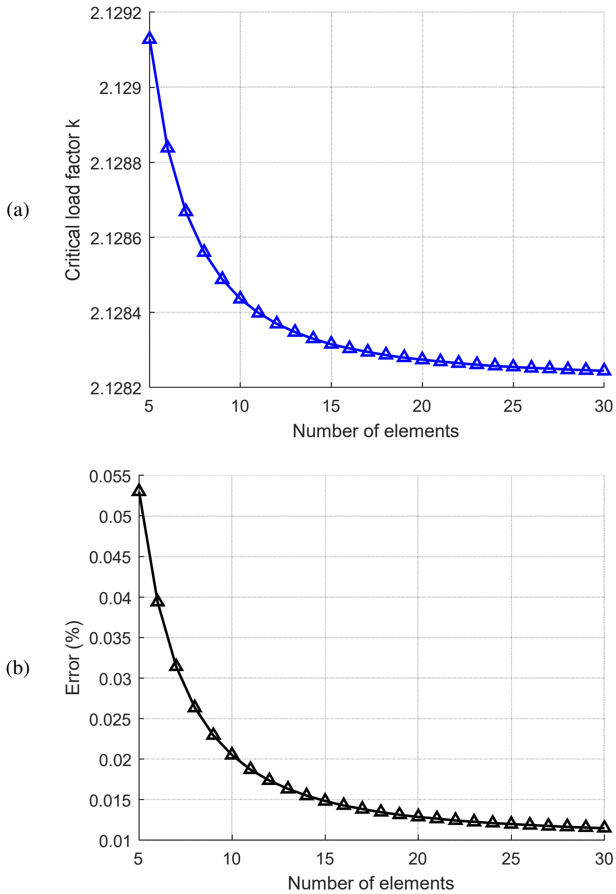


Fig. 5. Convergence of the critical force and number of elements when the stiffness ratio is α = 0.6. (a) Stability coefficient *k*, (b) error.

The stability coefficient *k,* calculated using the finite element method, is illustrated in Figures 4(a) and 5(a). The calculation results reveal an adequate convergence of the finite element method. High accuracy was achieved by dividing the column into only four elements. As shown in Figures 4(b) and 5(b) the error of the finite element method was negligible.

B. *Example 2*

Consider the two-boundary condition of a column with stiffness *EI* changing according to a quadratic law, resting on an elastic foundation with stiffness K_{dh}. The boundary conditions at the two ends of the hinged-hinged column (Case 1) and the clamped-free column (Case 2) are subjected to a concentrated force at the end of the column (Figure 6).

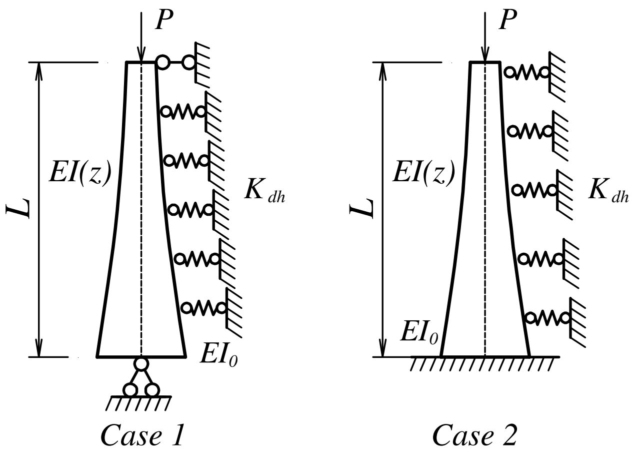


Fig. 6. Column subjected to a concentrated load.

In this example, the finite element solution is implemented by dividing the column into elements of equal length.

We introduce the normalized critical force to evaluate the influence of the elastic foundation:

$$k_{dh} = \frac{K_{dh} L^3}{EI_0} \tag{15}$$

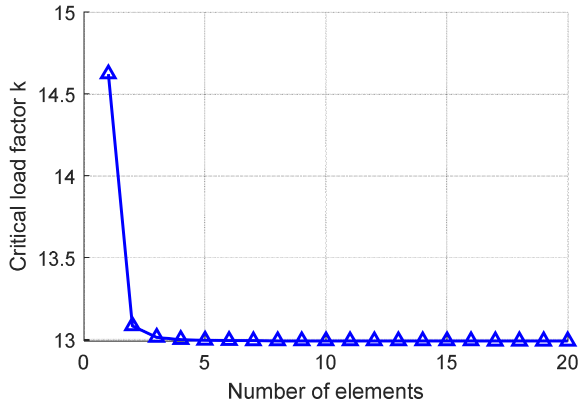
In this example, the bending stiffness of the column follows the following rule:

$$EI = EI_0 \left(1 - 0.2\frac{z}{L}\right)^2 \tag{16}$$

In this example, the column is divided into 1 to 20 elements to investigate the convergence of the critical force *k* with the geometric length factor *λ*. Figure 7 shows the results of the convergence of the critical load factor. Figure 8 presents the results of the convergence of the geometric length factor of the hinged-hinged column and the clamped-free column, considering an elastic foundation stiffness of $k_{dh} = 10$. Figures 7 and 8 show that the critical force and geometric length factor converge quickly when there are five or more elements.

Figures 9 and 10 display the critical force coefficient and the geometric length coefficient in relation to the elastic foundation stiffness. The results revealed that when the foundation stiffness increases, the structure's stiffness increases, the critical force increases, and the geometric length coefficient decreases. Figure 9 shows that in the hinged-hinged column (Case 1), the critical force increases almost linearly with the elastic foundation, whereas in the clamped-free column (Case 2), the critical force increases more slowly than the increase in stiffness of the elastic foundation. Moreover, the geometric length coefficient (Figure 10) decreases when the stiffness of the foundation increases.

(a)



(b)

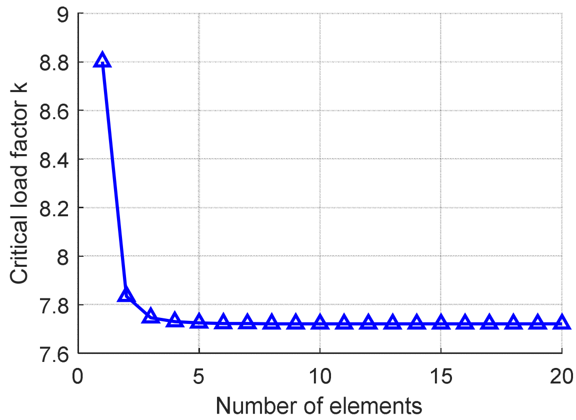
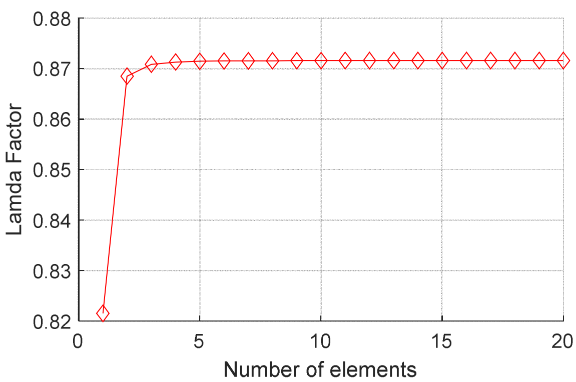


Fig. 7. Randomness in the elastic modulus of the beam. (a) Case 1, (b) Case 2.

(a)



(b)

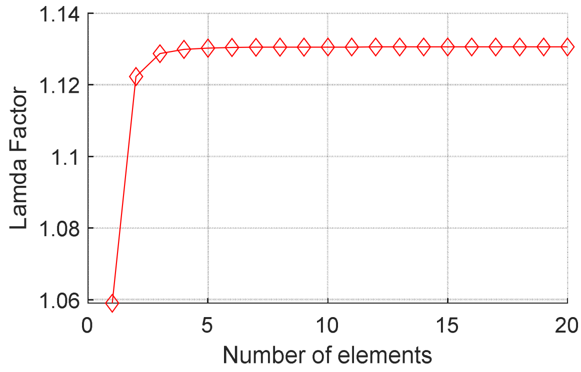


Fig. 8. Randomness in the elastic modulus of the beam. (a) Case 1, (b) Case 2.

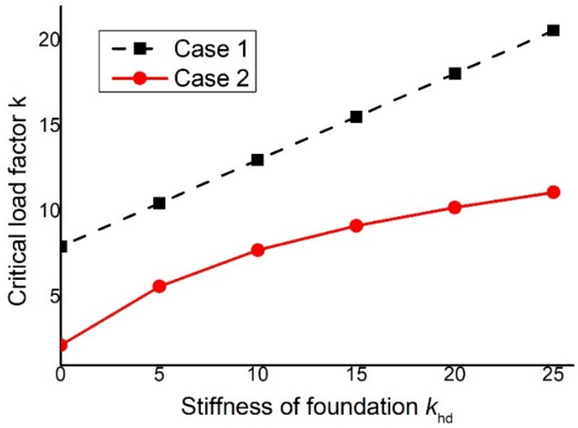


Fig. 9. Column stability coefficient k and stiffness of the elastic foundation.

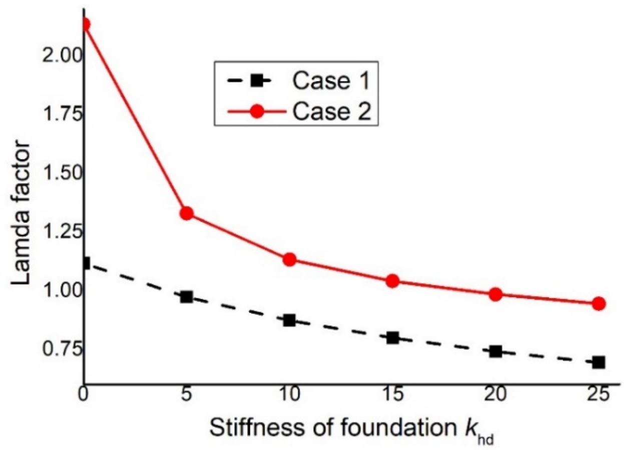


Fig. 10. Geometric length coefficient according to the elastic foundation stiffness.

IV. CONCLUSIONS

This paper presents the results of the use of the finite element method in calculating the stability of a column, resting on an elastic foundation, with a variable cross-section. The bending stiffness matrix for a beam with a variable cross-section and the stiffness matrix for an elastic foundation were established. Algorithms and computational programs were developed in MATLAB. Numerical examples showing the results of the critical load calculation demonstrate that the finite element method when compared to the analytical method yields highly accurate results. The numerical examples included various elastic foundations of different stiffnesses for the hinged-hinged column and the clamped-free column to assess the influence of these parameters on the column's critical force and geometric length parameters. The calculation results reveal that when the foundation's stiffness increases, the critical force increases and the geometric length coefficient decreases. Furthermore, the critical force of a hinged-hinged column increases faster than that of a clamped-free column as the stiffness of the elastic foundation rises. The finite element formulas in this paper can be applied to many types of column cross-sections with different shapes, including uniformly changing cross-sections and step-shaped cross-sections.

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