Limit and Shakedown Analysis of Thin Plates under Random Strength by Probabilistic Constrained Programming

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ABSTRACT

This work presents a new model for the shakedown analysis of Kirchhoff plates under uncertain conditions of the plastic moment by the direct method. The stochastic models of the plastic moment are normal or lognormal distribution. New formulations are derived to compute the lower bound and upper shakedown loads and a dual algorithm is established to calculate the upper and lower bound shakedown load factors simultaneously for a chosen structural reliability level. An example is examined to illustrate the algorithm and shows robust results of the stochastic analysis.

Keywords-plates; limit load; shakedown; reliability; stochastic programming; chance constraints

I. INTRODUCTION

Thin structures in the form of plates and shells are encountered in many technical fields such as civil engineering, mechanical engineering, and aeronautical, marine, and chemical industries. In this paper, we consider the bending of plates subjected to lateral loads. Limit analysis of plates in bending for design against plastic collapse has been started around 50 year ago. Authors in [1, 2] are considered to be the first to publish works on limit analysis of plate bending. Fox [1] used the slip line method to obtain the exact solution for clamped plate loaded by uniform pressure. Hodge and Belytschko [2] formulated the problem of limit analysis of plates as an optimization problem using the finite element method. Shakedown analysis of plates and shells for design against progressive and alternating plasticity has been developed in [3]. In recent years, several publications have presented algorithms to solve large problems with millions of variables and overcome the size limitation of general nonlinear programming [4-7]. Shakedown analysis is an extension of limit analysis in which applied loads vary with time in a load domain. A dual algorithm was proposed in [4] to solve the deterministic problem of lower bound and upper bound shakedown loads simultaneously for plate bending. Limit and shakedown reliability analysis of thin shells with uncertain strength and loading has been presented for the calculation of failure probabilities in [8, 9].

As a more direct stochastic programming approach developed for continuum finite elements in [10], we reformulate a similar algorithm as the deterministic equivalent of a chance constrained program in which the lower bound and upper bound limit and shakedown load of a plate under uncertain strength is computed. If the thickness is deterministic and the yield stress is normally or lognormally distributed, a deterministic equivalent formulation can be derived which allows a most effective numerical calculation of load limits for a prescribed reliability or failure probability of the structure.

II. THE STATIC APPROACH TO PROBABILISTIC CONSTRAINED PROGRAMMING

In the static approach we look for the maximum of a safe load in an admissible moments field. The admissible moments field satisfies the static conditions of equilibrium and plastic admissibility. Melan's static theorem states that a structure will shakedown, if there exists a time-variant elastic moment field $\mathbf{m}^E(\mathbf{x},t)$ and a time-independent residual moment field $\bar{\mathbf{m}}(\mathbf{x})$

such that the yield condition is satisfied for any loading path at any time t and in any point $\mathbf{x}$ of the plate [11, 12].

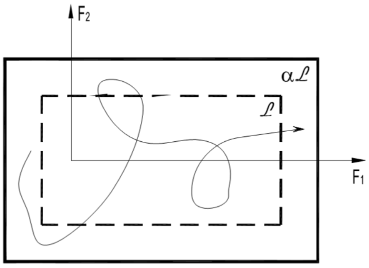

Fig. 1. Convex load domain $\mathcal{L}$ for two forces.

The maximum increase of the load domain $\mathcal{L}$ for a plate made of elastic, perfectly plastic material up to $\alpha^{-}\mathcal{L}$, which still allows shakedown, characterized by the load factor α^{-} can be obtained by solving the following optimization problem, in FEM formulation:

$$\alpha^{-} = \max \alpha$$
$$\text{s.t.:}\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\mathbf{m}}_i = 0 \\ f(\alpha \mathbf{m}_{ik}^E + \bar{\mathbf{m}}_i) \le m_0 \end{cases} \quad (1)$$
$$\forall i = \overline{1, NG} \quad \forall k = \overline{1, n}$$

where $\mathbf{B}_i$ is the deformation matrix, w_i is integration weight at Gauss point i and NG denotes the total number of Gauss points of the structure. The first constraint in (1) is the equilibrium equation of residual moments. The second constrained is the yield condition. Due to the convexity of the load domain $\mathcal{L}$ in Figure 1, the inequality has to be checked only at the n load vertices so that the problem becomes time-independent. Shakedown occurs for any load history in $\alpha^{-}\mathcal{L}$. Limit analysis is the special case of monotonic loading for which the inequality of the yield condition has to be checked only for one load point, $n = 1$. The limit load is independent of the elastic data and residual stress, which disappears at plastic collapse.

Now we consider the situation that the plastic moment of the plate is not given but must be modelled by a random variable $m_0 = m_0(\omega)$ on a certain probability space. Under uncertainty of the plastic moment, the inequalities in (1) are not always satisfied, the i^{th} yield condition is required to be satisfied with a probability greater than some reliability level ψ_i. In most applications, the same reliability ψ is chosen for all $i = \overline{1, NG}$ to achieve a desired failure probability $P_f = \psi - 1$ of the structure. Problem (1) must be reformulated with probabilistic constraints:

$$\alpha^{-} = \max \alpha$$
$$\text{s.t.:}\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\mathbf{m}}_i = \mathbf{0} \\ \text{Prob}\left[f(\alpha \mathbf{m}_{ik}^E + \bar{\mathbf{m}}_i) - m_{0i}(\omega) \le 0 \right] \ge \psi_i \end{cases} \quad (2)$$

After some transformations, the probabilistic constrained problem (2) can be converted into the equivalent deterministic problem (3) or (4). If the plastic moment of the plate $m_{0i}(\omega)$ is distributed normally with mean μ_i and standard deviation σ_i, the stochastic problem (2) can be convert into the equivalent deterministic problem:

$$\alpha^{-} = \max \alpha$$
$$s.t.:\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\mathbf{m}}_i = \mathbf{0} \\ f(\alpha \mathbf{m}_{ik}^E + \bar{\mathbf{m}}_i) \le \mu_i - \kappa \sigma_i \end{cases} \quad (3)$$

If the plastic moment of the plate $m_{0i}(\omega)$ is distributed lognormally with parameters μ_i and σ_i, the equivalent deterministic problem is:

$$\alpha^{-} = \max \alpha$$
$$s.t.:\begin{cases} \sum_{i=1}^{NG} w_i \mathbf{B}_i^T \bar{\mathbf{m}}_i = \mathbf{0} \\ f(\alpha \mathbf{m}_{ik}^E + \bar{\mathbf{m}}_i) \le e^{\mu_i - \kappa \sigma_i} \end{cases} \quad (4)$$
$$\forall i = \overline{1, NG} \quad \forall k = \overline{1, n}$$

III. KINEMATIC APPROACH TO PROBABILISTIC CONTRAINED PROGRAMMING

The kinematic approach is based on Koiter's kinematic theorem [12, 13]. The shakedown load factor can be found by searching for the minimum of the plastic dissipation of a plate in a kinematic velocity field of curvatures $\dot{\boldsymbol{\kappa}}$ [24]. The upper bound of the shakedown problem can be expressed as the following convex nonlinear programming problem, in FEM formulation:

$$\alpha^{+} = \min \sum_{k=1}^{n} \sum_{i=1}^{NG} w_i m_0 \sqrt{\dot{\boldsymbol{\kappa}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\kappa}}_{ik}}$$
$$\text{s.t.:}\begin{cases} \sum_{k=1}^{n} \dot{\boldsymbol{\kappa}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \quad \forall i = \overline{1, NG} \\ \sum_{k=1}^{n} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\kappa}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^E = 1 \end{cases} \quad (5)$$

where:

$$\mathbf{Q} = \begin{bmatrix} 4/3 & 2/3 & 0 \\ 2/3 & 4/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}. \quad (6)$$

and the displacement field within a DKQ element of the plate can be expressed in terms of its nodal values $\mathbf{u} = \left[w \quad \partial w / \partial x \quad \partial w / \partial y \right]^{\mathrm{T}}$.

If the yield stress of the material is random, then the plastic moment is an uncertain quantity and the objective function of (5) is a stochastic variable. The problem (5) must now be reformulated as a stochastic program [15]:

$$\min \rho$$
$$\text{s.t.:} \begin{cases} \text{Prob}\left(\sum_{k=1}^{n} \sum_{i=1}^{NG} w_i m_{0i}(\omega) \sqrt{\dot{\boldsymbol{\kappa}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\kappa}}_{ik}} \geq \rho \right) = \psi \\ \sum_{k=1}^{n} \dot{\boldsymbol{\kappa}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{n} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\kappa}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{7}$$

Depending on whether the distribution of the random variable is normal or lognormal, (7) can be converted into equivalent deterministic problems (8) and (9), respectively.

- The equivalent deterministic program for normal distribution of strength:

$$\alpha^{+} = \min \sum_{k=1}^{n} \sum_{i=1}^{NG} w_i (\mu_i - \kappa \sigma_i) \sqrt{\dot{\boldsymbol{\kappa}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\kappa}}_{ik} + \varepsilon^2}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{n} \dot{\boldsymbol{\kappa}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{n} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\kappa}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{8}$$

- The equivalent deterministic program for lognormal distribution of strength:

$$\alpha^{+} = \min \sum_{k=1}^{n} \sum_{i=1}^{NG} w_i e^{\mu - \kappa \sigma} \sqrt{\dot{\boldsymbol{\kappa}}_{ik}^{\mathrm{T}} \mathbf{Q} \dot{\boldsymbol{\kappa}}_{ik} + \varepsilon^2}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{n} \dot{\boldsymbol{\kappa}}_{ik} = \mathbf{B}_i \dot{\mathbf{u}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{n} \sum_{i=1}^{NG} w_i \dot{\boldsymbol{\kappa}}_{ik}^{\mathrm{T}} \mathbf{m}_{ik}^{E} = 1 \end{cases} \tag{9}$$

The objective functions of nonlinear constrained optimization problems (8), (9) are not everywhere differentiable. This issue is overcome by a "smooth regularization method", in which a small positive number ε^2 is added to the objective functions [16].

IV. THE ALGORITHM FOR SOLUTIONS

For convenience, some new variables are defined:

$$\begin{aligned} \dot{\mathbf{x}}_{ik} &= w_i \mathbf{Q}^{1/2} \dot{\boldsymbol{\kappa}}_{ik}, \\ \mathbf{m}_{ik} &= (\mathbf{Q}^{-1/2})^{T} \mathbf{m}_{ik}^{E}, \\ \hat{\mathbf{B}}_i &= w_i \mathbf{Q}^{1/2} \mathbf{B}_i \end{aligned} \tag{10}$$

Note that:

$$\begin{aligned} &\mathbf{Q}^{1/2} \mathbf{Q}^{-1/2} = \mathbf{I}, \\ &\left(\mathbf{Q}^{1/2} \right)^{T} \mathbf{Q}^{1/2} = \mathbf{Q} \end{aligned} \tag{11}$$

Inserting the new variables in (10) into (8) and (9) gives a shorter version of the primal problem:

$$\alpha^{+} = \min \sum_{k=1}^{n} \sum_{i=1}^{NG} R \sqrt{\dot{\mathbf{x}}_{ik}^{T} \dot{\mathbf{x}}_{ik} + \varepsilon^2}$$
$$\text{s.t.:} \begin{cases} \sum_{k=1}^{n} \dot{\mathbf{x}}_{ik} = \hat{\mathbf{B}}_i \dot{\mathbf{u}} \qquad \forall i = \overline{1, NG} \\ \sum_{k=1}^{n} \sum_{i=1}^{NG} \dot{\mathbf{x}}_{ik} \mathbf{m}_{ik}^{T} = 1 \end{cases} \tag{12}$$

In (12), $R = \mu_i - \kappa \sigma_i$, $R = e^{\mu_i - \kappa \sigma_i}$ for the cases of normal strength and lognormal strength, respectively.

By the duality theory, we can prove that the maximum problem:

$$\alpha^{-} = \max_{\alpha, \boldsymbol{\beta}_i} \alpha$$
$$\text{s.t.:} \begin{cases} \left\| \boldsymbol{\beta}_i + \alpha \mathbf{m}_{ik} \right\| \leq R \\ \sum_{i=1}^{NG} \hat{\mathbf{B}}_i^{T} \boldsymbol{\beta}_i = \mathbf{0} \end{cases} \tag{13}$$

is the dual of the minimum problem (12). Problem (13) can be written with the von Mises yield function as follows after some transformations [10]:

$$\alpha^{-} = \max_{\alpha, \boldsymbol{\beta}_i} \alpha$$
$$\text{s.t.:} \begin{cases} f\left[\alpha \mathbf{m}_{ik}^{E} + \bar{\mathbf{m}}_i \right] \leq R \\ \sum_{i=1}^{NG} \hat{\mathbf{B}}_i^{T} \boldsymbol{\beta}_i = \mathbf{0} \end{cases} \tag{14}$$

The problem (14) is exactly in the form of the lower bound shakedown limit, which is formulated in (4). The problem (13) is identical to problem (14). This shows that problem (14) is a dual problem of (12).The duality of the convex optimization problems lets the lower and upper bound converge to the same load factor.

As in (8) and (9), the objective function of (12) is not everywhere differentiable. Therefore, a small positive number ε^2 is added to the objective function. The first and second constraint of (12) are eliminated with a penalty method, and then the Lagrange multipliers method is employed to convert (12) into an unconstrained programming problem. The Lagrange-penalty function of the problem is:

$$F_p = \sum_{i=1}^{NG} \left[\begin{array}{l} \sum_{k=1}^{n} R \sqrt{\dot{\mathbf{x}}_{ik}^{T} \dot{\mathbf{x}}_{ik} + \varepsilon^2} \\ + \frac{c}{2} \left(\sum_{k=1}^{n} \dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right)^{T} \left(\sum_{k=1}^{n} \dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} \right) \end{array} \right] \tag{15}$$

where c is a penalty parameter ($c \gg 1, \varepsilon \approx 10^{10}$).

The corresponding Lagrange function of (12) is:

$$L = F_p - \alpha\left(\sum_{i=1}^{NG}\sum_{k=1}^{n}\dot{\mathbf{x}}_{ik}^T\mathbf{m}_{ik} - 1\right) \tag{16}$$

We denote:

$$\boldsymbol{\beta}_i = -c\left(\sum_{k=1}^{n}\dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i\dot{\mathbf{u}}\right) \tag{17}$$

The optimally conditions for problem (12) now become

$$\begin{cases} \dfrac{\partial L}{\partial \dot{\mathbf{x}}_{ik}} = -\boldsymbol{\beta}_i - \alpha\mathbf{m}_{ik} + R\left(\dfrac{\dot{\mathbf{x}}_{ik}}{\sqrt{\dot{\mathbf{x}}_{ik}^T\dot{\mathbf{x}}_{ik} + \varepsilon^2}}\right) = \mathbf{0} & (a) \\ \dfrac{\partial L}{\partial \dot{\mathbf{u}}} = \boldsymbol{\beta}_i^T\hat{\mathbf{B}}_i = \mathbf{0} & (b) \\ \sum_{k=1}^{n}\dot{\mathbf{x}}_{ik} = \hat{\mathbf{B}}_i\dot{\mathbf{u}} & (c) \\ \sum_{k=1}^{n}\sum_{i=1}^{NG}\dot{\mathbf{x}}_{ik}\mathbf{m}_{ik}^T = 1 & (d) \end{cases} \tag{18}$$

The following algorithm performs an approximation to solve the system of the nonlinear equations in (18) by using Newton's method and making use of (13).

Step 1. Creating a starting point of the velocity and the curvature rate vectors $\dot{\mathbf{u}}^0, \dot{\mathbf{x}}^0$ such that the normalized condition (18d) is satisfied:

$$\sum_{i=1}^{NG}\sum_{k=1}^{m}\mathbf{m}_{ik}^T\dot{\mathbf{x}}_{ik}^0 = 1 .$$

Choose values for the penalty parameter c and for ε and set the vector $\boldsymbol{\beta}_i^0$ equal to zero: $\boldsymbol{\beta}_i^0 = \mathbf{0}$.

Step 2. Compute the increment of the nodal value vector $d\dot{\mathbf{u}}$, the curvature rate $d\dot{\mathbf{x}}$, and $(\alpha + \mathrm{d}\,\alpha)$ by solving the system of the equations:

$$\begin{cases} \mathbf{H}_{ik}d\dot{\mathbf{e}}_{ik} + \sqrt{\dot{\mathbf{x}}_{ik}^T\dot{\mathbf{x}}_{ik} + \varepsilon^2}\left[c\left(\sum_{k=1}^{m}d\dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i d\dot{\mathbf{u}}\right) - d\alpha\mathbf{m}_{ik}\right] = -\mathbf{f}_{ik} \\ c\hat{\mathbf{B}}_i^T\left(\sum_{k=1}^{m}d\dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i d\dot{\mathbf{u}}\right) = -\mathbf{h}_i \\ \sum_{i=1}^{NG}\sum_{k=1}^{m}\mathbf{m}_{ik}^T d\dot{\mathbf{x}}_{ik} = 1 - \sum_{i=1}^{NG}\sum_{k=1}^{m}\dot{\mathbf{x}}_{ik}^T\mathbf{m}_{ik} \end{cases}$$

in which:

$$\begin{cases} \mathbf{H}_{ik} = R\mathbf{I}_{ik} + \left[c\left(\sum_{k=1}^{n}\dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i\dot{\mathbf{u}}\right) + \alpha\mathbf{m}_{ik}\right]\dfrac{\dot{\mathbf{x}}_{ik}^T}{\sqrt{\dot{\mathbf{x}}_{ik}^T\dot{\mathbf{x}}_{ik} + \varepsilon^2}} \\ \mathbf{f}_{ik} = R\dot{\mathbf{x}}_{ik} + c\sqrt{\dot{\mathbf{x}}_{ik}^T\dot{\mathbf{x}}_{ik} + \varepsilon^2}\left(\sum_{k=1}^{m}\dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i\dot{\mathbf{u}}\right) - \alpha\sqrt{\dot{\mathbf{x}}_{ik}^T\dot{\mathbf{x}}_{ik} + \varepsilon^2}\mathbf{m}_{ik} \\ \mathbf{h}_i = c\hat{\mathbf{B}}_i^T\left(\sum_{k=1}^{m}\dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i\dot{\mathbf{u}}\right) \end{cases}$$

Step 3. Perform a line-search to find the step size λ_k such as:

$$F_P\left(\dot{\mathbf{u}} + \lambda d\dot{\mathbf{u}},\ \dot{\mathbf{x}} + \lambda d\dot{\mathbf{x}}\right) \rightarrow \min .$$

Update the displacement and the curvature rate as:

$$\begin{aligned} \dot{\mathbf{u}} &= \dot{\mathbf{u}} + \lambda_k d\dot{\mathbf{u}}, \\ \dot{\mathbf{x}}_{ik} &= \dot{\mathbf{x}}_{ik} + \lambda_k d\dot{\mathbf{x}}_{ik} \end{aligned} .$$

Step 4. Compute the increment of the vector:

$$d\boldsymbol{\beta}_i = -c\left(\sum_{k=1}^{m}d\dot{\mathbf{x}}_{ik} - \hat{\mathbf{B}}_i d\dot{\mathbf{u}}\right) - \mathbf{h}_i$$

Perform a line search to find the step size λ_s such that:

$$\lambda_k = \max\lambda$$
$$s.t.: \left\|\boldsymbol{\beta}_i + \alpha\mathbf{m}_{ik} + \lambda\left(d\boldsymbol{\beta}_i + \mathbf{m}_{ik}d\alpha\right)\right\| \le R$$

Update the vector $\boldsymbol{\beta}_i$ and load factor α:

$$\boldsymbol{\beta}_i = \boldsymbol{\beta}_i + \lambda_s d\boldsymbol{\beta}_i ,$$

$$\alpha = \alpha + \lambda_a d\alpha .$$

Step 5. Check the convergence criteria:

- $\left|\alpha^+(iter) - \alpha^+(iter-1)\right| \le$ precision
- $\left|\alpha^-(iter) - \alpha^-(iter-1)\right| \le$ precision

If they are satisfied then stop, otherwise repeat steps 2-4.

V. NUMERICAL EXAMPLES

In this example, we calculate the limit load for a rectangular as shown in Figure 2. The data are the short length $b = 5\,\mathrm{m}$, $a = 2b$, the plate thickness $t = 0.1\,\mathrm{m}$, the mean value of the yield stress $E(\sigma_0) = 250\,\mathrm{MPa}$ and the standard deviation $\sigma = 0.1E(\sigma_0)$. With $m_p = \sigma_0 t^2/4$ the mean value of the plastic moment (per length) is $\mu = E(m_p) = 0.625N$ and its standard deviation is $\sigma = 0.1E(m_p)$. The reliability level $\psi = 0.999$ is chosen. Let us consider this plate for two cases: the simply supported plate and the clamped plate. The plate is modelled by 1024 DKQ elements. The limit loads for both cases are compared with those found in other studies some in Tables I and II. The analytical lower bound for the simply supported plate ($a = 2b$) derived by Johansen [17]:

$$\begin{aligned} p_L &= 8\left(1 + \frac{a}{b} + \frac{b}{a}\right)\cdot\frac{m_p}{ab} = \\ &= 8\left(1 + 2 + \frac{1}{2}\right)\cdot\frac{m_p}{2b^2} = 28\frac{m_p}{2b^2} \end{aligned} \tag{19}$$

and the upper bound by Ingerslev [18] for $b < a$:

$$p_L = \frac{24}{\left(\sqrt{3+\left(\frac{b}{a}\right)^2} - \frac{b}{a}\right)^2} \cdot \frac{m_p}{b^2} = \tag{20}$$

$$= \frac{24}{\left(\sqrt{3+0.5^2} - 0.5\right)^2} \cdot \frac{m_p}{b^2} = 28.28\frac{m_p}{2b^2}$$

are used in Table I.

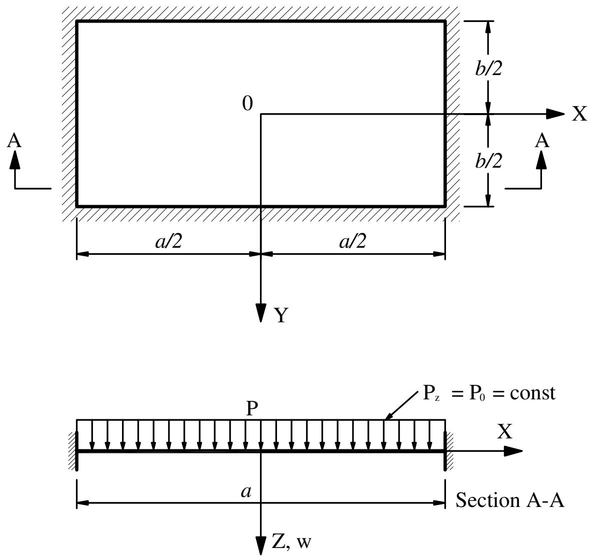

Fig. 2. Rectangular plate subjected to uniform load.

TABLE I. LIMIT LOAD FACTOR COMPARISON FOR SIMPLY SUPPORT PLATE ($\times \frac{m_p}{ab}$)

Reference	Lower bound	Upper bound	Stochastic strength model
[17]	28		Analytic, deterministic
[18]		28.28	Analytic, deterministic
[6]	–	29.88	Deterministic
[19]	–	29.88	Deterministic
Present	30.535	30.615	Deterministic
Present	21.11	21.15	Normal
Present	22.22	22.38	Lognormal

TABLE II. LIMIT LOAD FACTOR COMPARISON FOR CLAMPED SUPPORT PLATE ($\times \frac{m_p}{ab}$)

Reference	Lower bound	Upper bound	Stochastic strength model
[6]	–	54.61	Deterministic
Present	55.91	55.91	Deterministic
Present	38.58	38.58	Normal
Present	40.86	40.86	Lognormal

Equations (19)-(20) were used to diagram the limit loads over the failure probability in Figure 4. If the deterministic upper bound (20) is used, we get:

$$\alpha_{\lim} = 28.28 \cdot \sigma_0 \frac{t^2}{4 \cdot 2b^2} = 28.28\frac{m_p}{2b^2},$$

and then the normal limit load factor is:

$$\alpha_{\lim} = 28.28(1-\kappa \cdot \nu)\sigma_0 \frac{t^2}{4 \cdot 2b^2} = 28.28(1-\kappa \cdot \nu)\frac{m_p}{2b^2}$$

and the lognormal limit load factor is:

$$\alpha_{\lim} = 28.28\frac{e^{(\mu-\kappa\sigma)}}{\sigma_0}\frac{t^2}{4 \cdot 2b^2} = 28.28\frac{e^{(\mu-\kappa\sigma)}}{\sigma_0}\frac{m_p}{2b^2}.$$

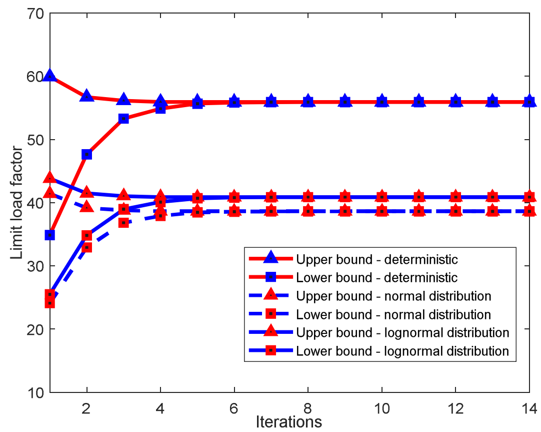

Fig. 3. Convergence of the limit load factors ($\times \frac{m_p}{ab}$).

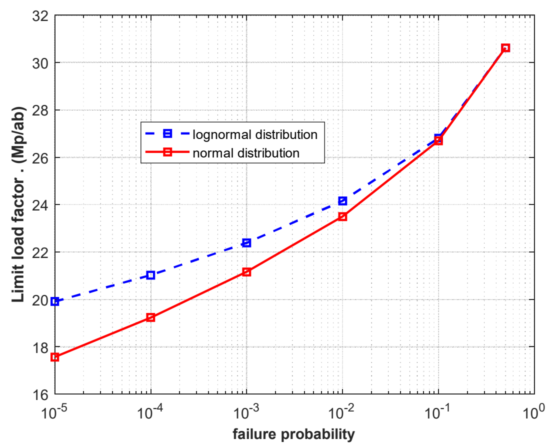

Fig. 4. Limit load factor over failure probability for simply plate ($\times \frac{m_p}{ab}$).

Figure 4 shows that the limit load factor must be reduced strongly even to achieve a low reliability of the structure. In structural reliability the failure probability P_f is computed for the limit load factor α of a given structure. P_f is very sensitive to variations of the stochastic model. If the stochastic model is changed between a normal and a lognormal distribution, a small P_f changes for more than an order of magnitude on the logarithmic scale at fixed α as seen in Figure 4.

The presented chance constraint analysis computes α for a chosen P_f . Figure 4 shows that α changes only a little between the stochastic models. Therefore, the proposed new stochastic approach is much more robust than the established structural reliability analysis.

VI. CONCLUSION

This paper presents the so-called probabilistic constrained programming method to perform limit and shakedown analyses of plates under random material strength. This stochastic programming method reformulates the optimization problem as a deterministic equivalent of the stochastic limit or shakedown analysis problem. Uncertainties can be quantified and a target failure probability is chosen according to the failure consequences. Then a design load of plates can be calculated on the basis of the stochastic model and the data of all uncertainties. The method is robust and has the same numerical effort as a deterministic limit and shakedown analysis. In the presented numerical tests, the method converges in only 4-5 iterations, which means the effort of 4-5 linear elastic analyses. This is highly effective compared to fully plastic reliability analyses possibly over a many load cycles or a complex load history.

The results of the survey for rectangular plates in this paper were compared with the results of published works and can be considered reliable. In the case of deterministic plate strength, the proposed dual algorithm gives limit load results similar to those suggested by other authors. Moreover, the algorithm can also find solutions for the lower and upper bound problems simultaneously. In the case of the plastic moment of the plate being a random variable with normal or logarithmic distribution, the results obtained in this work are new and have not been published in any previous documents. These results show that if we consider the random condition of the plastic moment (yield stress of material), the limit load factor of the plates decreases strongly.

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