Research on the Application of the Model Order Reduction Algorithm in Designing a Robust Controller for the Balance System of a Self-Balancing Two-Wheeled Bicycle

Ngo Kien Trung
Hanoi Industrial Textile Garment University, Hanoi, Vietnam
trungnk@hict.edu.vn

Nguyen Thi Tham
Technical and Economic College, Thai Nguyen, Vietnam
nguyenthamcdktkt@gmail.com

Trinh Thi Diep
Technical and Economic College, Thai Nguyen, Vietnam
ttdiepk15d1@gmail.com

Vu Thi Anh Ngoc
Technical and Economic College, Thai Nguyen, Vietnam
v.t.a.ngoc@gmail.com

Hong Quang Nguyen
Thai Nguyen University of Technology, Thai Nguyen, Vietnam
quang.nguyenhong@tnut.edu.vn (corresponding author)

Received: 19 November 2024 | Revised: 7 December 2024, 18 December 2024, and 26 December 2024 | Accepted: 29 December 2024

 DOI: https://doi.org/10.48084/etasr.9649

ABSTRACT

This paper focuses on the design analysis and control of a Self-Balancing Two-Wheeled Bicycle (SBTWB) model. The difficulty of the two-wheeled bicycle balance control problem is that the two-wheeled bicycle model is uncertain and is continuously affected by disturbances. Many different control methods have been proposed to design an SBTWB balance controller, but the most suitable algorithm is the robust control algorithm. However, the robust controller of an SBTWB is often complex and of higher order, which affects the quality of the control process. This study introduces a Model Order Reduction (MOR) algorithm based on the preserving dominant poles and applies this algorithm to simplify the 15th order robust controller of the balance control system of an SBTWB. Through comparison and evaluation, it is shown that the 5th-order controller or the 4th-order controller can replace the 15th-order robust controller. Through a simulation of the control system using the 4th-order controller, it is demonstrated that the proposed 4th-order controller ensures a stable balance of the SBTWB, while the 4th-order controllers according to other order reduction methods cannot maintain the balance of the SBTWB. The simulation results show the effectiveness of the order-reduction algorithm based on the conservation of dominant pole points and the robust control algorithm for the SBTWB.

Keywords-self-balancing two-wheeled bicycle; model order reduction; robust control algorithm; dominant poles; high-order controller

I. INTRODUCTION

Among the means of transport, two-wheeled vehicles, including motorbikes and electric bicycles have the advantages of low fuel consumption, fast acceleration, flexibility, narrow parking space, and low cost, and are especially suitable for narrow road conditions in large cities. However, the biggest limitation of two-wheeled vehicles is that they cannot balance themselves like cars. Drivers of two-wheeled vehicles do not have the same comfort as when driving a car. Simultaneously, when colliding with another vehicle, a conventional two-wheeled vehicle will fall and the driver could get injured. Therefore, it is necessary to develop a two-wheeled vehicle that can balance itself while standing still, when moving, and when colliding, creating comfort for the driver similar to that of a car. If a self-balancing two-wheeled vehicle is well designed, it will only be thrown out when it collides and will still maintain its vertical position thanks to the self-balancing system installed on it, thus ensuring the safety of the user. However, it is difficult to control the system to achieve self-balance under different working conditions and when the load changes. The self-balancing of two-wheeled vehicles is attractive from both theoretical and practical perspectives, thus research on self-balancing two-wheeled vehicles has attracted the attention of many scientists as well as manufacturing companies.

Research on balance control of the SBTWB has been conducted since the late 20th century. Basically, there are three balance control methods for two-wheeled vehicles:

(i) Controlling balance using a flywheel, as presented in [1-7].

(ii) Controlling balance by centrifugal force [8].

(iii) Controlling balance by changing the center of gravity [9].

Among these three methods, controlling the balance using a flywheel has the advantage of being responsive and working even when the vehicle is not moving.

Authors in [1] built a model of an SBTWB using a gyroscope system and used control algorithms to balance two-wheeled vehicles. The balancer consisted of two flywheels rotating in two opposite directions and was designed with three control loops: steering angle control to change the direction of movement, vehicle speed control, and balance control. The controllers were implemented using an 80C196KC microcontroller with a clock frequency of 20 Mhz. Author in [5] designed a model of an SBTWB using a flywheel, operating based on the principle of gyroscopes to balance the vehicle. The two-wheeled vehicle was built deploying the Lagrange method and the controller was based on the root trajectory method. The experimental results showed that the controller could rapidly maintain vehicle balance. In [7], a model of an SBTWB was developed using a flywheel operating on the principle of gyroscopes to balance the vehicle. The bicycle balance controller was designed using the robust control method. The simulation and experimental results exhibited that the controller could maintain the balance of the bicycle when it was stationary and moving. Authors in [2-4] proposed an SBTWB model using a flywheel, based on the inverted pendulum principle. However, a PID controller was utilized for the bicycle balance system [3, 4]. The simulation and experimental results in [3] demonstrated that the bicycle model can balance when it is stationary, moving straight, or moving along a curve.

Self-balancing bicycle models using flywheels based on the gyroscope principle [1, 5-7] often dissipate a large amount of energy because the flywheel must rotate at a high speed to create a balancing torque for the bicycle. If a bicycle is powered by a limited energy source, such as a battery, then dissipating too much energy to maintain the balance of the bicycle will limit the operating time of the latter. On the contrary, self-balancing bicycle models using flywheels based on the inverted pendulum principle [2-4] often dissipate little energy, because the flywheel typically rotates at a low speed to create a balancing torque for the bicycle. If the bicycle is powered only by a limited energy source, such as a battery, with a low energy level to maintain balance, the bicycle model based on this principle will have a long operating time and save energy. Because an SBTWB often must operate under different conditions, the carrying capacity may vary, and the external forces acting on the vehicles may change. It is difficult to find the model of an SBTWB, and a two-wheel bicycle can be considered as indeterminate objects/an indeterminate object [5]. Owing to the uncertain nature of the two-wheeled vehicle model, among the control algorithms proposed to control the SBTWB, such as nonlinear control [1, 9], compensation design using the original trajectory approach [5], PD control [6], the robust control [7] is the most suitable one for controlling an uncertain object. However, the control design using the robust control method H_∞ [10] often leads to a controller with a high order. A high-order controller complicates the control program, increases the calculation time of the control system, and slows the system response. The disadvantages of a high-order control system may cause the control system to fail to meet the requirements of real-time control, causing the vehicle to lose its balance. Therefore, reducing the order of the controller while still ensuring its quality is important in practical applications. Two methods can be used to reduce the controller order. The first method entails optimization algorithms to determine the parameters of a low-order controller (pre-selected) such that the low-order controller meets the requirements of the sustainable control problem. The second method consists of designing the controller in two steps. In the first step, the controller is designed according to a robust control algorithm to obtain a high-order controller. In the second step, the high-order controller is reduced according to the order-reduction algorithm to obtain a reduced-order controller.

According to [7], in the first method, the controller can be a low-order controller, but two optimization problems must be solved simultaneously, namely problems in finding the parameters of the controller and robust control. This leads to difficulties with this method. The parameters of the low-order controller may not be determined if the chosen controller is not suitable. In the second method, the order reduction problem is an independent problem, so it always gives the order reduction result, as in [11]. For this reason, the second method has an advantage over the first method because a low-order controller can be found in any scenario.

This study proposes a control method for a two-wheeled bicycle using a model reduction algorithm in two steps:

- Design the H_∞ controller to control the balance of a two-wheeled bicycle, which is called a full-order controller.
- Apply an order-reduction algorithm to reduce the order of the full-order controller to the lower-order controller while ensuring quality.

II. THE DYNAMIC MODEL OF SELF-BALANCING TWO-WHEELED BICYCLE

The SBTWB model is designed based on the principle of balance using a flywheel according to the inverted pendulum principle [2-4]. To maintain the balance of the vehicle, this flywheel rotates around the axis (with an acceleration $\dot{\varphi}$) and creates torque to compensate for the torque created by the gravity of the vehicle. To control the acceleration $\dot{\varphi}$ of the flywheel, a DC motor was used. The speed and acceleration of the motor (or flywheel) change when the voltage U supplied to the motor changes. Therefore, the problem of controlling the balance of a two-wheeled bicycle becomes a problem of maintaining the tilt angle of the vehicle at 0° by controlling the voltage U supplied to the DC motor. The SBTWB that was built is shown in Figure 1.

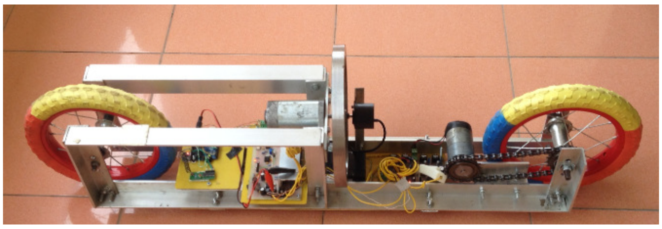

Fig. 1. The SBTWB model.

The technical parameters of the model were as follows: the length of the model was 1.19 m. The height of the model was 0.5 m. The width of the model was 0.4 m. The diameter of the flywheel was 0.26 m. The mass of the flywheel was 3.976 kg. The technical parameters of the control system included a DC motor rotating the flywheel 100 W–15V–3400 rpm, a flywheel motor speed sensor Encoder Sharo 100 pulse, and an altitude angle sensor GY-521 MPU-6050. The forward and backward movements of the model were controlled using a DC motor.

III. THE MATHEMATICAL MODEL OF SELF-BALANCING TWO-WHEELED BICYCLE

The dynamic model of the SBTWB is illustrated in Figure 2, where m_1 is the bicycle weight (including the DC motor), m_2 is the flywheel weight, h_1 is the height of the center of gravity of the bicycle (excluding the flywheel), h_2 is the height of the center of gravity of the flywheel, I_1 is the inertial torque of the bicycle, I_2 is the inertial torque of the flywheel, θ is the tilt angle of the bicycle corresponding to the vertical axis, φ is the rotation angle of the flywheel.

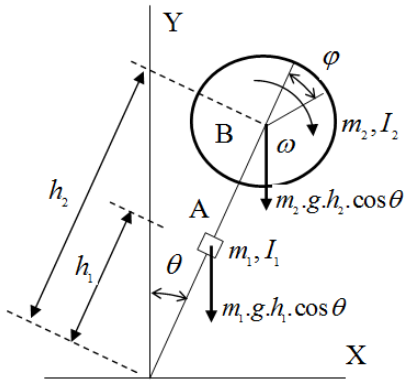

Fig. 2. Dynamic model of the SBTWB.

To build the dynamic model of the system, the Lagrange equation was used [5]:

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{q}_i}\right\} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \tag{1}$$

where T is the total kinetic energy of the system, V is the total potential energy of the system, Q_i is the external force, and q_i is the general coordinate system.

The results of building the dynamic model of the system are:

$$(m_1h_1^2 + m_2h_2^2 + I_1 + I_2)\ddot{\theta} + I_2\ddot{\varphi} - g \cdot \sin\theta\,(m_1h_1 + m_2h_2) = 0 \tag{2}$$

$$I_2\ddot{\varphi} + I_2\ddot{\theta} = aK_m\left[\frac{U - K_e\dot{\varphi}}{R}\right] \tag{3}$$

Considering a DC motor with a gear ratio of a:1, K_m is the motor torque constant, K_e is the back-emf constant, and R is the resistance of the motor.

When the vehicle is in actual operation, the tilt angle of the vehicle is very small ($\theta < 10^0$), it can be considered ($\theta = \varphi \approx 0$, $\sin\theta \approx \theta$). With the condition $\theta = \varphi \approx 0$, $\sin\theta \approx \theta$, (2) can be linearized into the following form:

$$(m_1h_1^2 + m_2h_2^2 + I_1 + I_2)\ddot{\theta} + I_2\ddot{\varphi} - g \cdot \theta(m_1h_1 + m_2h_2) = 0 \tag{4}$$

Taking $x = \begin{bmatrix} \theta = x_1 \\ \dot{\theta} = x_2 \\ \dot{\varphi} = x_3 \end{bmatrix}$ as the state variable, $y = \theta, u = U$ and $\boldsymbol{A} = (m_1h_1^2 + m_2h_2^2 + I_1 + I_2)$; $\boldsymbol{B} = (m_1h_1 + m_2h_2)$, we obtain the state-space model as follows:

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u$$
$$y = \boldsymbol{C}x + \boldsymbol{D}u \tag{5}$$

with:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{Bg}{(A-I_2)} & 0 & \frac{aK_mK_e}{R(A-I_2)} \\ -\frac{Bg}{(A-I_2)} & 0 & -aK_mK_e\frac{A}{I_2R(A-I_2)} \end{bmatrix}$$

$$\boldsymbol{B}=\begin{bmatrix}0\\-\frac{aK_m}{R(A-I_2)}\\aK_m\frac{A}{I_2R(A-I_2)}\end{bmatrix}$$

$$\boldsymbol{C}=[1\quad 0\quad 0]$$

$$\boldsymbol{D}=[0]$$

The technical parameters of the SBTWB model are displayed in Table I.

TABLE I. THE PARAMETERS OF THE SBTWB MODEL

Parameter	Value	Unit
I_1	0.1404	Kg·m^2
h_1	0.15	m
I_2	0.03289	Kg·m^2
h_2	0.205	m
m_1	10.024	Kg
m_2	3.976	Kg
K_e	0.04215	V·s/Rad
K_m	0.04215	Nm/A
R	0.267	Ω
a	1:1	
g	9.81	m/s^2

Substituting the values of the parameters in Table I into the system of equations (5), the following results are obtained:

$$\boldsymbol{A}=\begin{bmatrix}0 & 1 & 0\\42.6718 & 0 & 0.0125\\-42.6718 & 0 & -0.2148\end{bmatrix}$$

$$\boldsymbol{B}=\begin{bmatrix}0\\-0.2962\\5.0959\end{bmatrix}$$

$$\boldsymbol{C}=[1\quad 0\quad 0]$$

Converting system (5) into a transfer function form, we get:

$$\mathbf{O}(s)=\frac{\theta(s)}{U(s)}=\frac{-0.2962s-3.288\cdot10^{-17}}{s^3+0.2148s^2-42.67s-8.633} \tag{6}$$

IV. ROBUST CONTROLLER FOR SELF-BALANCING TWO-WHEELED BICYCLE

Regarding the process of modeling an SBTWB, it can be observed that when the SBTWB is in actual operation, the bicycle can carry additional loads; therefore, the height of the SBTWB's center of gravity will change, and the SBTWB 's mass will change. When the SBTWB moves, the environmental factors affecting the vehicle, such as the external forces and obstacles on the road, can also change.

Additionally, the uncertainty of some technical parameters of the SBTWB and the environment will lead to an uncertainty in the vehicle model. Uncertain factors can reduce the accuracy of a vehicle's mathematical model, thereby reducing the control quality, and can even make the control system unstable. Therefore, to ensure the requirement of stable vehicle control, robust control is suitable for the two-wheeled vehicle balance control system [10]. A robust control system stabilizes the product quality, regardless of the changes in the object as well as the disturbances affecting the system. The purpose of the robust control is to maintain closed-loop quality despite changes in the object. The structure of the robust control system for the SBTWB is depicted in Figure 3.

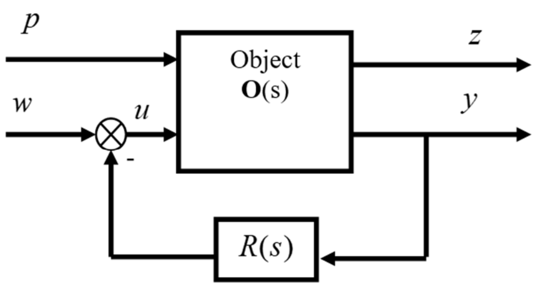

Fig. 3. Structure diagram of the robust control system.

Where **O**(s) is the model of the controlled object, **R**(s) is the controller, *p* is the unwanted signal acting on the system, *z* is the unwanted output, *w* is the set signal, *y* is the system output, and *u* is the control signal.

To determine the robust controller **R**(s), the Youla parameterization algorithm was followed [11, 12] and the following results were obtained:

$$\mathbf{R}(s)=\frac{\mathbf{H}(s)}{\mathbf{D}(s)} \tag{7}$$

with:

$$\mathbf{H}(s)=1.26s^{15}+110.4s^{14}+3959s^{13}+8.089\cdot10^4s^{12}+1.078\cdot10^6s^{11}+1.006\cdot10^7s^{10}+6.869\cdot10^7s^9+3.547\cdot10^8s^8+1.419\cdot10^9s^7+4.478\cdot10^9s^6+1.116\cdot10^{10}s^5+2.142\cdot10^{10}s^4+2.96\cdot10^{10}s^3+2.616\cdot10^{10}s^2+1.183\cdot10^{10}s+1.536\cdot10^9$$

$$\boldsymbol{D}(s)=-0.0001185s^{15}-0.02028s^{14}-1.051s^{13}-22.89s^{12}-222.5s^{11}-165.6s^{10}+1.999\cdot10^4s^9+2.433\cdot10^5s^8+1.533\cdot10^6s^7+5.942.10^6s^6+1.438\cdot10^7s^5+2.042\cdot10^7s^4+1.401\cdot10^7s^3+2.108\cdot10^6s^2+1.49\cdot10^{-8}s$$

V. MODEL ORDER REDUCTION FOR ROBUST CONTROLLER OF SELF-BALANCING TWO-WHEELED BICYCLE

The 15th order controller (7) will lead to many disadvantages in implementing SBTWB balance control because the complex program code will increase the processing time, the response speed of the control system will be slow, it will not meet the real-time requirements of the controller, and it can render the balance control system unstable. Therefore, it is necessary to reduce the order of the 15th-order controller to simplify the control program, reduce the processing time of the controller, and increase the response speed of the control system while still satisfying the requirements for sustainable stability of the control system.

Algorithms for determining low-order mathematical models from high-order mathematical models that satisfy certain basic requirements, such as preserving stability and small order reduction errors, form a field called MOR.

A. The Model Order Reduction Problem

Over the years, hundreds of studies have been published and proposed to solve the MOR problem of high-order models, most of which focus on solving the problem of order reduction for linear systems.

Consider the state-space model of a linear continuous system, constant parameters, many inputs, and many outputs as:

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u$$
$$y = \boldsymbol{C}x \tag{8}$$

where:

$x \in \boldsymbol{R}^n, u \in \boldsymbol{R}^p, y \in \boldsymbol{R}^q, \boldsymbol{A} \in \boldsymbol{R}^{nxn}, \boldsymbol{B} \in \boldsymbol{R}^{nxp}, \boldsymbol{C} \in \boldsymbol{R}^{qxn}$

The task of the MOR algorithms is to determine a low-order model in the following form:

$$\dot{x}_r = \boldsymbol{A}_r x_r + \boldsymbol{B}_r u$$
$$y_r = \boldsymbol{C}_r x_r \tag{9}$$

where:

$x_r \in \boldsymbol{R}^r, u_r \in \boldsymbol{R}^p, y_r \in \boldsymbol{R}^q, \boldsymbol{A}_r \in \boldsymbol{R}^{rxr}, \boldsymbol{B}_r \in \boldsymbol{R}^{rxp}, \boldsymbol{C}_r \in \boldsymbol{R}^{qxr}$ and $r \ll n$.

The low-order model (9) can replace the high-order model (8) in the simulation and control problems.

B. The Model Order Reduction Algorithm Based on Preserving Dominant Poles

In the authors' opinion, the "best" reduction algorithm, that is, a reduction method that meets all requirements, does not exist. Each algorithm has its own advantages and disadvantages, and should be used according to the appropriate requirements. With the goal of applying the reduction algorithm to the problem of reducing the order of a high-order controller, the reduction algorithm must ensure a small reduction error and high computational efficiency. At the same time, because the high-order controller is an unstable model, the reduction algorithm needs to be able to reduce the order for both stable and unstable models. However, in reality, most proposed reduction algorithms are mainly applied to stable models [13, 14], and few studies have been conducted on unstable models.

There are two basic methods for MOR of an unstable system. The first method is an indirect order reduction algorithm. This algorithm divides the unstable original system into stable and unstable components and then applies the order reduction algorithm to the stable components [15-23]. Finally, to obtain the order of reduction of the root system, the reduced stable components are added to the unstable components. The second method (direct order reduction algorithm) modifies and adjusts the order-reduction algorithms so that the order reduction can be performed regardless of whether the original system is stable or unstable [24-29].

In this study, a reduction algorithm is introduced based on the dominant pole preservation method [16], which can reduce the order of both stable and unstable models, as described below:

<u>Input</u>: The system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ of (8) (unstable system).

<u>Step 1</u>: The unstable system is decomposed into stable and unstable subsystems.

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{d11} & 0 \\ 0 & \boldsymbol{A}_{d22} \end{bmatrix}; \ \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{d1} \\ \boldsymbol{B}_{d2} \end{bmatrix}; \ \boldsymbol{C} = [\boldsymbol{C}_{d1} \quad \boldsymbol{C}_{d2}]$$

where:

$\boldsymbol{A}_{d11} \in \boldsymbol{R}^{mxm}, \boldsymbol{A}_{d22} \in \boldsymbol{R}^{(n-m)x(n-m)}, \boldsymbol{B}_{d1} \in \boldsymbol{R}^{mxp}, \boldsymbol{B}_{d2} \in \boldsymbol{R}^{(n-m)xp}, \boldsymbol{C}_{d1} \in \boldsymbol{R}^{qxm}, \boldsymbol{C}_{d2} \in \boldsymbol{R}^{qx(n-m)}$

and *m* is the number of the stable poles, and *m-n* is the number of the unstable poles.

Stable subsystem $(\boldsymbol{A}_{d11}, \boldsymbol{B}_{d1}, \boldsymbol{C}_{d1})$.

Unstable subsystem $(\boldsymbol{A}_{d22}, \boldsymbol{B}_{d2}, \boldsymbol{C}_{d2})$.

<u>Step 2</u>: Triangle realization of the stable subsystem $(\boldsymbol{A}_{d11}, \boldsymbol{B}_{d1}, \boldsymbol{C}_{d1})$ in which $\boldsymbol{A}_{d11}$ has a triangle form, according to the following steps:

Step 2.1: Compute the Schur decomposition of $\boldsymbol{A}_{d1}$: $\boldsymbol{A}_{d1} = \boldsymbol{U\Delta U}^T$, where $\boldsymbol{U}$ is a unitary matrix and $\boldsymbol{\Delta}$ is an upper triangle matrix.

Step 2.2: Solve the following Lyapunov equation to determine the observability Gramian $\boldsymbol{Q}$.

$$\boldsymbol{\Delta Q} + \boldsymbol{Q\Delta} + (\boldsymbol{CU})^T(\boldsymbol{CU}) = 0$$

Step 2.3: Decompose the Cholesky gramian of the observation $\boldsymbol{Q}$ to determine the upper triangular matrix $\boldsymbol{R}$.

$$\boldsymbol{Q} = R^T\boldsymbol{R}$$

Step 2.4: Compute nonsingular transformation $\boldsymbol{T} = \boldsymbol{UR}^{-1}$.

Step 2.5: Compute $(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}) = (\boldsymbol{T}^{-1}\boldsymbol{A}_{d1}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{B}_{d1}, \boldsymbol{C}_{d1}\boldsymbol{T})$.

<u>Step 3</u>: Re-order the poles on the main diagonal of the upper – triangle matrix $\widetilde{\boldsymbol{A}}$ using the $\boldsymbol{H}_\infty$- dominant index.

Input: The $(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{C}})$ system.

Step 3.1: For each pole λ_i, $i = 1, .. n$ compute its $\boldsymbol{H}_\infty$- dominant index $\boldsymbol{R}_i = \frac{\|\tilde{C}_i\tilde{B}_i\|_2}{|Re\ \lambda_i|}$.

Step 3.2: Choose the largest $\boldsymbol{H}_\infty$- dominant index.

Step 3.3: Reorder the pole λ_{i_1} (and its conjugate $\bar{\lambda}_{i_1}$, if it appears) to the first position in the diagonal of $\widetilde{\boldsymbol{A}}$ by unitary matrix $\mathbf{U}_1$:

$$U_1^*\tilde{A}U_1 = \left[\begin{array}{cc|ccc} \lambda_{i_1} & * & * & * & * \\ & \bar{\lambda}_{i_1} & * & * & * \\ \hline & & * & * & * \\ & & & \ddots & * \\ & & & & * \end{array}\right].$$

Step 3.4: Compute the new equivalent realization $(\boldsymbol{U}_1^T\widetilde{\boldsymbol{A}}\boldsymbol{U}_1, \boldsymbol{U}_1^T\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}\boldsymbol{U}_1)$.

Step 3.5: Remove the first two rows and columns of $\left(\boldsymbol{U}_1^T\widetilde{\boldsymbol{A}}\boldsymbol{U}_1, \boldsymbol{U}_1^T\widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{C}}\boldsymbol{U}_1\right)$ to obtain a smaller realization $\left(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{B}}, \widehat{\boldsymbol{C}}\right)$ with $n-2$ dimensions.

Step 3.6: Perform the same procedure from Step 3.1 to Step 3.5 for smaller realizations $\left(\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{B}}, \widehat{\boldsymbol{C}}\right)$ and continue this loop until all poles are re-ordered.

Output*:* The equivalent system $\left(\breve{\boldsymbol{A}}, \breve{\boldsymbol{B}}, \breve{\boldsymbol{C}}\right)$ with the poles that are arranged in descending $\boldsymbol{H}_\infty$-dominant indices on the main diagonal of the upper – triangle matrix $\breve{\boldsymbol{A}}$.

Step 4: Reduce $\left(\breve{\boldsymbol{A}}, \breve{\boldsymbol{B}}, \breve{\boldsymbol{C}}\right)$ system.

Input step 4: The equivalent system $\left(\breve{\boldsymbol{A}}, \breve{\boldsymbol{B}}, \breve{\boldsymbol{C}}\right)$.

Step 4.1: Choose re-order r so that $r \ll n$.

Step 4.2: $\left(\breve{\boldsymbol{A}}, \breve{\boldsymbol{B}}, \breve{\boldsymbol{C}}\right)$ is partitioned as follows:

$$\breve{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ 0 & \boldsymbol{A}_{22} \end{bmatrix}; \breve{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{bmatrix}; \breve{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix}$$

where $\boldsymbol{A_{11}} \in \boldsymbol{R}^{rxr}, \boldsymbol{B_1} \in \boldsymbol{R}^{rxp}, \boldsymbol{C_1} \in \boldsymbol{R}^{qxr}$.

Output step 4: The reduced order stable subsystem $(\boldsymbol{A}_{11}, \boldsymbol{B}_1, \boldsymbol{C}_1)$.

Output: The reduced order unstable system $\left((\boldsymbol{A}_{11}, \boldsymbol{B}_1, \boldsymbol{C}_1) + (\boldsymbol{A}_{d22}, \boldsymbol{B}_{d2}, \boldsymbol{C}_{d2})\right)$.

C. *Applied Model Order Reduction for Robust Controller of Self-Balancing Two-Wheeled Bicycle*

Applying the MOR algorithm based on preserving the dominant poles to reduce the order of the 15th order robust controller (8), the following results are obtained:

TABLE II. THE RESULTS OF THE ORDER REDUCTION OF THE 15TH ORDER ROBUST CONTROLLER

Order	Transfer function – (Rr(s))
5	$\frac{-1.063\cdot10^4s^5 - 4.383\cdot10^5s^4 - 2.877\cdot10^6s^3 - 6.463\cdot10^6s^2 - 2.638\cdot10^7s - 2.638\cdot10^7}{s^5 + 124.8s^4 + 2117s^3 - 3.62\cdot10^4s^2 - 3.797\cdot10^{-10}s - 1.666\cdot10^{-12}}$
4	$\frac{-1.063\cdot10^4s^4 - 6.758\cdot10^4s^3 - 2.353\cdot10^5s^2 - 7.301\cdot10^5s - 7.515.10^5}{s^4 + 89.69s^3 - 1031s^2 - 7.028\cdot10^{-12}s - 9.88\cdot10^{-14}}$
3	$\frac{-1.063\cdot10^4s^3 + 9.102\cdot10^4s^2 - 7226s - 7515}{s^3 - 10.31s^2 - 7.414\cdot10^{-14}s - 1.134\cdot10^{-15}}$

Rr(s) is the transfer function of the low-order robust controllers according to the model order reduction /MOR algorithm based on preserving dominant poles.

A simulation model of the SBTWB control system was built in Matlab-Simulink using the controllers evidenced in Table II. The results of the simulation are shown in Figure 4. Assuming that the bicycle is initially tilted at an angle $\theta = \pi/180$ (rad) from the vertical axis, the output tilt angle response of the SBTWB control system is depicted in Figures 5 and 6.

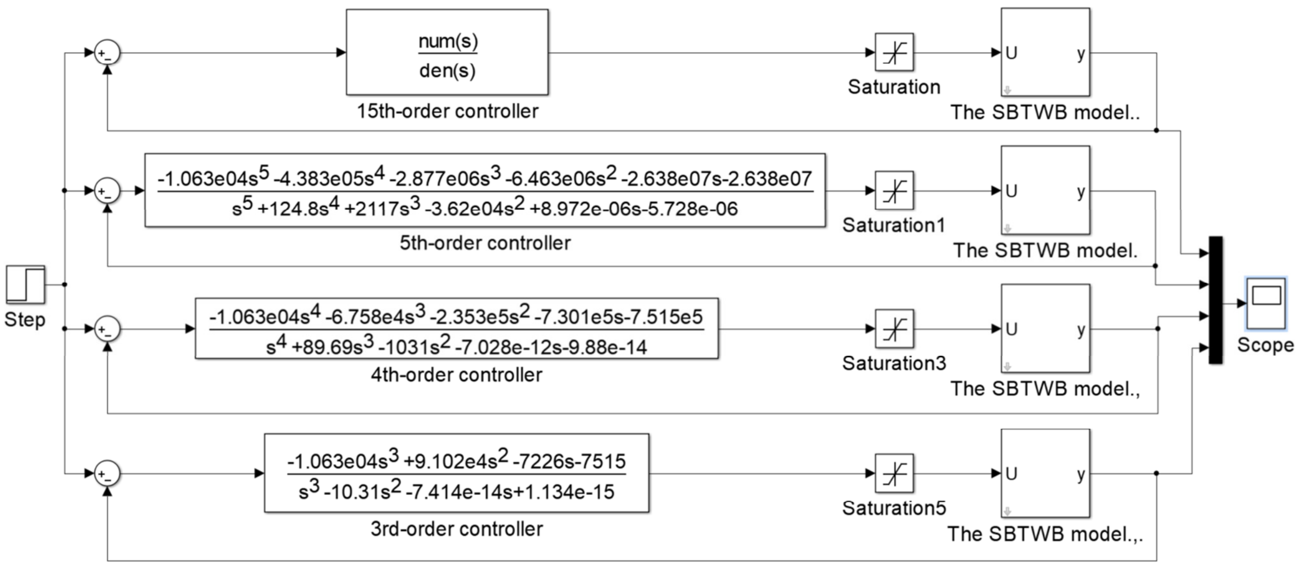

Fig. 4. The simulation structure diagram of the SBTWB control system.

As shown in Figure 5, the response of the control system using the 5th-order controller is almost identical to that of the control system using the 15th-order controller. The response of the control system using the 4th-order controller slightly deviates from the response of the control system using the original controller. When comparing the response quality parameters of the control system, the number of oscillations of the control system using the 4th-order controller (4) was larger than that of the control system using the 5th-order controller (3). The maximum overshoot value of the control system using the 4th order controller was larger than that of the control system using the 5th order controller. The control system utilizing the 5th- and 4th-order controllers is capable of maintaining a stable balance in the SBTWB model. However, the control system using a 3rd-order controller is not capable of maintaining a stable balance in the SBTWB model, as illustrated in Figure 6.

A 5th-order controller or 4th-order controller can be chosen to replace the 15th-order controller. The 5th-order controller is preferred when the output response quality of the system needs to be exactly the same as when the control system uses a 15th-order controller. The 4th-order controller should be selected when the controller is required to have the lowest possible order, while still ensuring the quality of the control system. In this study, a 4th-order controller instead of a 15th-order controller was chosen.

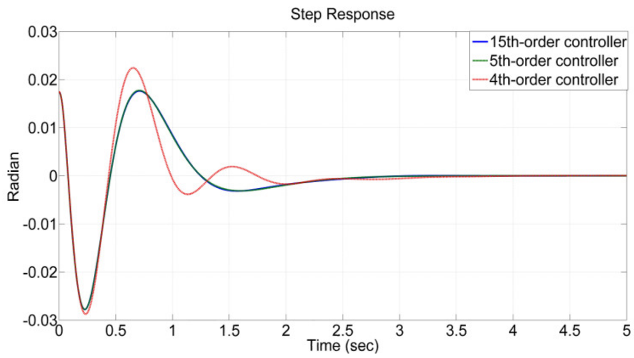

Fig. 5. The output tilt angle response of the control system using a 4th- and 5th-order controller and compared to the 15th-order controller.

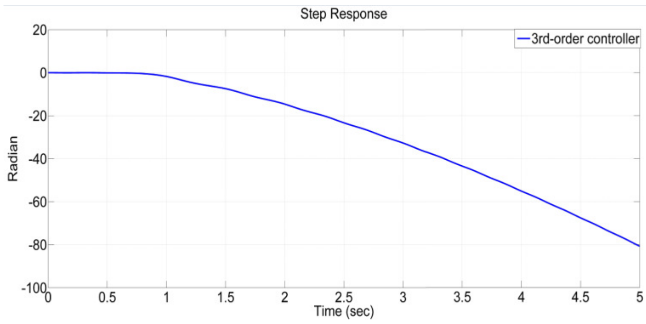

Fig. 6. The output tilt angle response of the control system using a 3rd-order controller.

In MOR, the most popular order reduction algorithms are the balanced truncation algorithm [30] and the stochastic balanced truncation algorithm based on Schur analysis [23, 24], which can be implemented in Matlab using the *balancmr* and *hankelmr* commands.

Applying the *balancmr* [30] command to the 15th-order controller, the following result is obtained:

$$R_{rb}(s) = \frac{-1.063\cdot10^4 s^4 - 7.235\cdot10^4 s^3 - 5.994\cdot10^5 s^2 + 4.402\cdot10^6 s - 2.259\cdot10^6}{s^4 + 90.37s^3 - 973.8s^2 - 18.92s + 0.0007323}$$

Applying the *Schurmr* [23, 24] command to the 15th-order controller, the following result is obtained:

$$R_{rc}(s) = \frac{-1.063\cdot10^4 s^4 - 7.243\cdot10^4 s^3 + 1.284\cdot10^7 s^2 - 1.507\cdot10^8 s + 3.99\cdot10^4}{s^4 + 90.44s^3 - 1039s^2 - 7.345\cdot10^{-12}s + 9.638\cdot10^{-28}}$$

Simulating the SBTWB control system in Matlab-Simulink using the 4th- order controllers according to the different order reduction algorithms, the results shown in Figure 7 are obtained. Assuming that the bicycle is initially tilted at an angle $\theta=\pi/180$(rad) from the vertical axis, the output tilt angle response of the SBTWB control system using the 4th-order controllers is displayed in Figure 8.

From Figure 8, it can be seen that the control system using the 4th-order controller according to the balance truncation algorithm cannot maintain a stable balance of the SBTWB model. Similarly, the control system using the 4th-order controller according to the Hankel optimization algorithm cannot maintain a stable balance. However, as observed in Figure 9, the control system using a 4th-order controller according to the MOR algorithm based on preserving the dominant poles can maintain a stable balance of the SBTWB model.

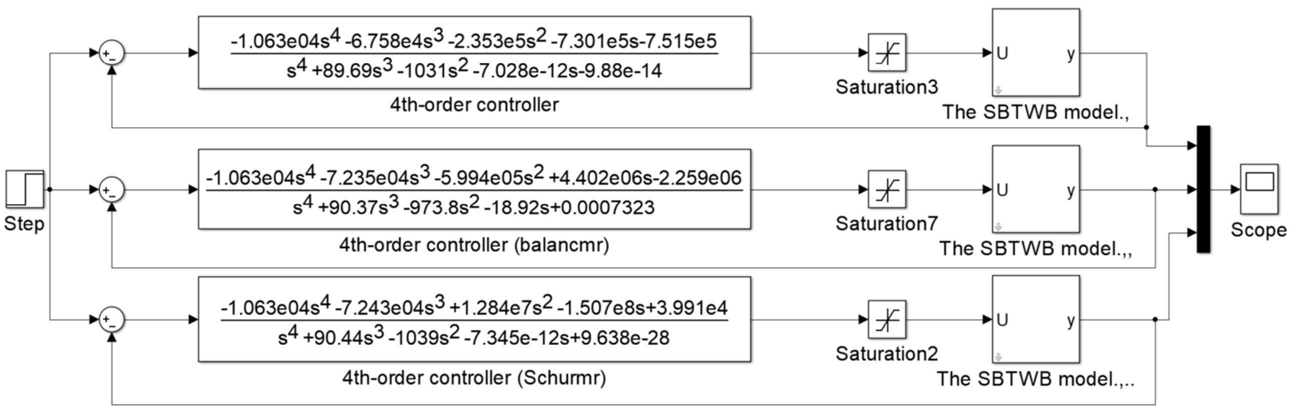

Fig. 7. The simulation structure diagram of the SBTWB control system using 4th-order controllers.

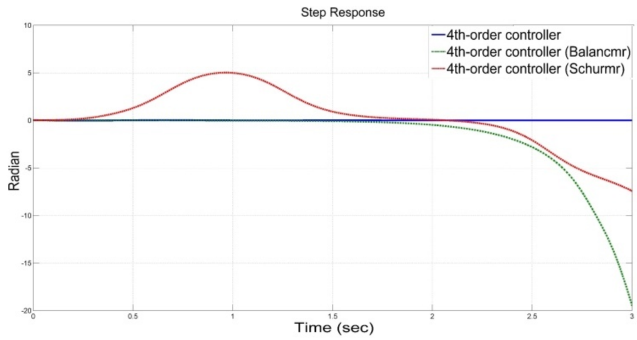

Fig. 8. The output tilt angle response of the control system using 4th-order controller.

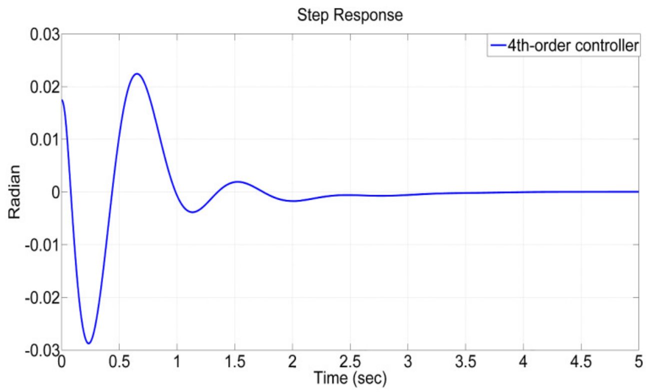

Fig. 9. The output tilt angle response of the control system using 4th-order controller according to the MOR algorithm based on preserving dominant poles.

VI. CONCLUSIONS

This study presents a Self-Balancing Two-Wheeled Bicycle (SBTWB) model. Through an uncertainty analysis of the SBTWB model, a robust controller was designed for the balance system of the SBTWB model and a 15th-order robust controller was obtained. A Model Order Reduction (MOR) algorithm based on preserving the dominant poles was also introduced and was applied to reduce the order of the 15th-order robust controller. After evaluating the reduced-order controllers, a 4th-order controller was chosen to replace the 15th-order robust controller. The results of comparing the effectiveness of the control system using the 4th-order controller according to different algorithms exhibit that the 4th-order controller according to the MOR algorithm based on preserving the dominant poles makes the SBTWB model stable, while the 4th-order controllers according to other order reduction algorithms make the SBTWB model unbalanced. Thus, for the specific problem, the MOR algorithm based on preserving the dominant poles is capable of solving the high-order controller order reduction problem better than the balanced truncation algorithm and the stochastic balanced truncation algorithm based on the Schur analysis.

The results of this study demonstrate the applicability of the MOR algorithm based on preserving the dominant poles in the order reduction problem of high-order controllers. In future studies, this algorithm will be applied to other order reduction problems.

ACKNOWLEDGMENT

The authors would like to thank the Thai Nguyen University of Technology, Vietnam, for their facilities and constructive criticism of the manuscript.

REFERENCES

[1] A. V. Beznos *et al.*, "Control of autonomous motion of two-wheel bicycle with gyroscopic stabilisation," in *Proceedings. 1998 IEEE International Conference on Robotics and Automation (Cat. No.98CH36146)*, Leuven, Belgium, May 1998, vol. 3, pp. 2670–2675, https://doi.org/10.1109/ROBOT.1998.680749.

[2] J.-X. Xu, A. Al Mamun, and Y. Daud, "Pendulum-balanced autonomous unicycle: Conceptual design and dynamics model," in *2011 IEEE 5th International Conference on Robotics, Automation and Mechatronics*, Qingdao, China, Sep. 2011, pp. 51–56, https://doi.org/10.1109/RAMECH.2011.6070455.

[3] S.-I. Lee, I.-W. Lee, M.-S. Kim, H. He, and J.-M. Lee, "Balancing and Driving Control of a Bicycle Robot," *Journal of Institute of Control, Robotics and Systems*, vol. 18, no. 6, pp. 532–539, 2012, https://doi.org/10.5302/J.ICROS.2012.18.6.532.

[4] Y. Kim, H. Kim, and J. Lee, "Stable control of the bicycle robot on a curved path by using a reaction wheel," *Journal of Mechanical Science and Technology*, vol. 29, no. 5, pp. 2219–2226, May 2015, https://doi.org/10.1007/s12206-015-0442-1.

[5] J. M. Gallaspy, "Gyroscopic stabilization of an unmanned bicycle," M.S Thesis, Auburn University, Auburn, USA, 1999.

[6] S. Suprapto, "Development of a gyroscopic unmanned bicycle," M.S Thesis, Asian Institute of Technology, Thailand, 2006.

[7] B. T. Thanh and M. Parnichkun, "Balancing Control of Bicyrobo by Particle Swarm Optimization-Based Structure-Specified Mixed H_2/H_∞ Control," *International Journal of Advanced Robotic Systems*, vol. 5, no. 4, Nov. 2008, Art. no. 39, https://doi.org/10.5772/6235.

[8] Y. Tanaka and T. Murakami, "Self sustaining bicycle robot with steering controller," in *The 8th IEEE International Workshop on Advanced Motion Control*, Kawasaki, Japan, Mar. 2004, pp. 193–197, https://doi.org/10.1109/AMC.2004.1297665.

[9] S. Lee and W. Ham, "Self stabilizing strategy in tracking control of unmanned electric bicycle with mass balance," in *IEEE/RSJ International Conference on Intelligent Robots and Systems*, Lausanne, Switzerland, Sep. 2002, vol. 3, pp. 2200–2205, https://doi.org/10.1109/IRDS.2002.1041594.

[10] D. McFarlane and K. Glover, "A loop-shaping design procedure using H/sub infinity / synthesis," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 759–769, Jun. 1992, https://doi.org/10.1109/9.256330.

[11] N. K. Vu and H. Q. Nguyen, "Design Low-Order Robust Controller for Self-Balancing Two-Wheel Vehicle," *Mathematical Problems in Engineering*, vol. 2021, no. 1, 2021, Art. no. 6693807, https://doi.org/10.1155/2021/6693807.

[12] D. Youla, H. Jabr, and J. Bongiorno, "Modern Wiener-Hopf design of optimal controllers–Part II: The multivariable case," *IEEE Transactions on Automatic Control*, vol. 21, no. 3, pp. 319–338, Jun. 1976, https://doi.org/10.1109/TAC.1976.1101223.

[13] H. H. Bui, "Control Design for the Ward–Leonard System in Wind Turbines," *Engineering, Technology & Applied Science Research*, vol. 13, no. 1, pp. 9968–9972, Feb. 2023, https://doi.org/10.48084/etasr.5425.

[14] D. S. Rao, M. S. Kumar, and M. R. Raju, "Design of Robust Controller for Higher Order Interval System using Differential Evolutionary Algorithm," *IAES International Journal of Robotics and Automation*, vol. 7, no. 4, pp. 232–250, Dec. 2018, https://doi.org/10.11591/ijra.v7i4.pp232-250.

[15] N. K. Vu and H. Q. Nguyen, "Model Order Reduction Algorithm Based on Preserving Dominant Poles," *International Journal of Control,*

Automation and Systems, vol. 19, no. 6, pp. 2047–2058, Jun. 2021, https://doi.org/10.1007/s12555-019-0990-8.

[16] K. Mustaqim, D. K. Arif, E. Apriliani, and D. Adzkiya, "Model reduction of unstable systems using balanced truncation method and its application to shallow water equations," *Journal of Physics: Conference Series*, vol. 855, no. 1, Jun. 2017, Art. no. 012029, https://doi.org/10.1088/1742-6596/855/1/012029.

[17] D. Novella Rodríguez, B. Del Muro Cuéllar, O. Sename, and M. Velasco Villa, "On the stabilization of high order systems with two unstable poles plus time delay," in *2012 20th Mediterranean Conference on Control & Automation (MED)*, Barcelona, Spain, Jul. 2012, pp. 12–17, https://doi.org/10.1109/MED.2012.6265607.

[18] C. S. Hsu and D. Hou, "Reducing unstable linear control systems via real schur transformation," *Electronics Letters*, vol. 27, no. 11, pp. 984–986, May 1991, https://doi.org/10.1049/el:19910614.

[19] S. K. Nagar and S. K. Singh, "An algorithmic approach for system decomposition and balanced realized model reduction," *Journal of the Franklin Institute*, vol. 341, no. 7, pp. 615–630, Nov. 2004, https://doi.org/10.1016/j.jfranklin.2004.07.005.

[20] N. K. Vu and H. Q. Nguyen, "Model reduction of unstable systems based on balanced truncation algorithm," *International Journal of Electrical and Computer Engineering*, vol. 11, no. 3, pp. 2045–2053, Jun. 2021, https://doi.org/10.11591/ijece.v11i3.pp2045-2053.

[21] Fatmawati, R. Saragih, R. T. Bambang, and Y. Soeharyadi, "Balanced truncation for unstable infinite dimensional systems via reciprocal transformation," *International Journal of Control, Automation and Systems*, vol. 9, no. 2, pp. 249–257, Apr. 2011, https://doi.org/10.1007/s12555-011-0206-3.

[22] K. Zhou, "Frequency weighted model reduction with $L\infty$ error bounds," in *1993 American Control Conference*, San Francisco, CA, USA, Jun. 1993, pp. 2123–2127, https://doi.org/10.23919/ACC.1993.4793256.

[23] M. G. Safonov and R. Y. Chiang, "Model Reduction for Robust Control: A Schur Relative-Error Method," in *1988 American Control Conference*, Atlanta, GA, USA, Jun. 1988, pp. 1685–1690, https://doi.org/10.23919/ACC.1988.4789991.

[24] E. Jonckheere and L. Silverman, "A new set of invariants for linear systems–Application to reduced order compensator design," *IEEE Transactions on Automatic Control*, vol. 28, no. 10, pp. 953–964, Oct. 1983, https://doi.org/10.1109/TAC.1983.1103159.

[25] H. H. Bui, "The Application of LQG Balanced Truncation Algorithm to the Digital Filter Design Problem," *Engineering, Technology & Applied Science Research*, vol. 12, no. 6, pp. 9458–9463, Dec. 2022, https://doi.org/10.48084/etasr.5235.

[26] K. Zhou, G. Salomon, and E. Wu, "Balanced realization and model reduction for unstable systems," *International Journal of Robust and Nonlinear Control*, vol. 9, no. 3, pp. 183–198, 1999, https://doi.org/10.1002/(SICI)1099-1239(199903)9:3<183::AID-RNC399>3.0.CO;2-E.

[27] A. Zilouchian, "Balanced structures and model reduction of unstable systems," in *IEEE Proceedings of the SOUTHEASTCON '91*, Williamsburg, VA, USA, Apr. 1991, pp. 1198–120, https://doi.org/10.1109/SECON.1991.147956.

[28] C. Boess, N. K. Nichols, and A. Bunse-Gerstner, "Model reduction for discrete unstable control systems using a balanced truncation approach." University of Reading, Preprint series, 2010.

[29] C. Boess, A. S. Lawless, N. K. Nichols, and A. Bunse-Gerstner, "State estimation using model order reduction for unstable systems," *Computers & Fluids*, vol. 46, no. 1, pp. 155–160, Jul. 2011, https://doi.org/10.1016/j.compfluid.2010.11.033.

[30] B. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE Transactions on Automatic Control*, vol. 26, no. 1, pp. 17–32, Feb. 1981, https://doi.org/10.1109/TAC.1981.1102568.